IDENTIFICATION OF INCOMPLETE INFORMATION GAMES IN MONOTONE EQUILIBRIUM

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Abstract. This paper develops identification results for the distribution of valuations in a class of allocation-transfer games. These games determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The results allow dependent valuations, discrete parts of the action space, non-differentiability, and unknown (to the econometrician) details of how the allocations and transfers are determined. The identification strategy is based on the assumption of monotone equilibrium, in which players use strategies that are weakly increasing functions of their valuations for the object being allocated.

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1. Introduction

This paper develops identification results for a class of allocation-transfer games that involve allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. Each of the players has a privately-known valuation for a unit of the object, and uses a strategy that relates its valuation to the action it takes in the game. The valuations can be dependent, including but not limited to "affiliated values." The identification result concerns recovering the distribution of these valuations from the data. The data corresponds to multiple instances ("plays") of the game. Partial identification results are stated in terms of "bounds" on the distribution of valuations in the sense of the usual multivariate stochastic order. Examples of allocation-transfer games include contests, auctions, public good provision, and various strategic market models.

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The identification strategy involves using the utility maximization problem to recover information about the unobserved valuation corresponding to an observed action. Hence, the identification strategy relates to an extensive literature in econometrics that uses utility maximization as a source of identification, particularly the use of optimality (e.g., "first order" or equilibrium) conditions in structural models. This approach is especially common in industrial organization, including (but not limited to) in models of firm behavior and monopoly/oligopoly (e.g., Rosse (1970), Bresnahan (1982), Lau (1982), Berry, Levinsohn, and Pakes (1995)) and models of auctions (e.g., Paarsch (1992), Donald and Paarsch (1993, 1996), Laffont, Ossard, and Vuong (1995), Guerre, Perrigne, and Vuong (2000), Athey and Haile (2002), and Aradillas-López, Gandhi, and Quint (2013)). These literatures have been reviewed in Berry and Tamer (2006), Paarsch and Hong (2006), Athey and Haile (2007), Berry and Reiss (2007), Reiss and Wolak (2007), Kline, Pakes, and Tamer (2021), and Kline and Tamer (2023) among other places.

However, beyond simply assuming utility maximization, the identification results are based on the assumption of monotone equilibrium. Each player uses a strategy that expresses its action as a function of its valuation. In a monotone equilibrium, the strategies are weakly increasing functions. Therefore, in a monotone equilibrium, if the valuation of a player increases then that player puts forth more effort in contest models, bids more in auction models, offers/demands more in market models, or contributes more in public good provision models. In addition to the intuitive appeal of monotone equilibrium, the economic theory literature has emphasized the importance of proving existence of monotone equilibrium in many specific games. Moreover, the economic theory literature, including Maskin and Riley (2000), Athey (2001), McAdams (2003, 2006), and Reny (2011), has also emphasized the importance of results that establish general conditions on the game that are sufficient for existence of monotone equilibrium. Therefore, the monotone equilibrium assumption can be motivated either as an intuitive assumption, or as a conclusion from the economic theory literature.

The identification result in this paper has multiple features. First, and most obviously, the identification result applies to a class of allocation-transfer games that involve allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. This class includes models of contests, auctions, procurement auctions and related models of oligopoly competition, partnership dissolution, public good provision, and strategic (non-"price taking") markets. The possible interpretations of the actions include effort in contest models, bids in auction models, bids/asks in market models, or contributions in public good provision models. In some games, as in auctions of a single unit, at most one player can be allocated a unit of the object. In other games, as in auctions of multiple units or public good provision, multiple players can be allocated a unit of the object. In some games, as in contests, the allocation can be non-deterministic. Therefore, the identification result can be viewed as exploring the identification power of the assumption of the use of monotone strategies across this entire class of allocation-transfer games.

Second, the identification strategy can handle the case of dependent valuations. Third, the identification strategy allows for discrete parts of the action space and non-differentiability. The action space can be discrete, continuous, or combinations of discrete and continuous. Allowing for dependent valuations and discrete actions combine to particularly complicate the identification problem. With discrete actions, necessarily a range of valuations use the same action (and this can happen also even without discrete actions, as discussed elsewhere in this paper), so those valuations cannot be distinguished based on observed behavior. This already complicates the identification problem, and should be expected to result in partial identification. Further, with dependent valuations, the utility maximization problem depends on the beliefs held by the player, which depend on the valuation of the player and therefore can be quite complicated, since players with different valuations have different beliefs about the valuations of the other players and hence different beliefs about the actions of the other players. The beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

Even with the goal of identification of valuations, it is important for the econometrician to recover some information about player's beliefs when using utility maximization as a source of identification, since beliefs are part of the mapping between valuations and observed behavior that results from utility maximization. The monotone equilibrium assumption allows a key step in the identification strategy whereby, essentially, the beliefs of a player who takes a given action can be shown to be suitably "bounded" between the beliefs of players who take larger and smaller actions.

Although continuity of the action space and differentiability is a common (simplifying) assumption in the related economic theory literature, discrete actions are common in empirical practice. For example, when the action is a monetary amount (e.g., a "bid" in an auction or "contribution" in public good provision), almost any realistic implementation in practice will place restrictions on the allowed bids. For instance, the implementation might require bids that are an integer multiple of some fixed amount (e.g., the allowed bids might be 5 dollars, 10 dollars, 15 dollars, etc.).[1] Discrete actions can also arise for other reasons. For instance, some public good provision models have a binary action space: contribute or not contribute, as in Example 5 in Appendix B. Lack of differentiability can arise even without discrete actions; see for instance Example 6 in Appendix B. This approach also accommodates fundamentally "non-numerical" actions, for example a binary "participation" decision when "participation" in the game is voluntary, as in some auction models in Example 2 in Appendix B.

Fourth, the identification strategy does not depend on the econometrician knowing the details of how the allocations and transfers are determined on the basis of the actions of the players, because it is possible to use the data to identify these objects. The equilibrium strategies that relate a player's valuation to that player's action will implicitly depend on these details. For example, the econometrician does not need to know the "contest success function" in models of contests, which relate the effort put forth by the players to the probabilities that each of them win the contest (see Example 1 in Appendix B). And for another example, the econometrician does not need to know the endogenous quantity function in auctions where the quantity of the object allocated depends on the actions of the players, as in a "supply curve" (see Example 2 in Appendix B). Such features of the game can be identified from the data, rather than assumed known, and the same identification strategy for the distribution of valuations applies regardless of the details of these features of the game.

[1] "Discrete" can be used with different definitions, which are worth distinguishing. Hortaçsu and McAdams (2010) studies an identification problem (and empirical application) in discriminatory price divisible goods auctions with independent private values. Kastl (2011) studies an identification problem (and empirical application) in uniform price divisible good auctions with (mainly) independent private values. In those models, bidders submit a bid function that specifies a quantity demanded for each possible price. Hence, neither model is covered by the allocation-transfer game framework studied in this paper, because those models deal with an action space that is a bid function rather than just a scalar bid. More importantly, the notion of "discrete" action is also different. In particular, Kastl (2011) uses "discrete" (per Kastl (2011, Assumption 3)) as a statement about the step function nature of the bid functions, where each player submits a bid function that is a step function, and therefore characterizable by a discrete vector of prices and quantities that characterize each "step" of the bid function. Hortaçsu and McAdams (2010) similarly emphasize step bid functions. However, the actual price and quantities at each step of the bid function is unrestricted. By contrast, as applied to auctions, this paper uses discrete as a statement on the restriction of the allowed bid levels. So, the players can only bid, for example, integer multiples of some minimal bid level. An earlier version of Hortaçsu (2002) looked at a model with a discrete grid of possible prices, and hence with a "discrete" action space more similar to the discreteness in this paper. Of course, the overall identification problem (and hence identification strategy) is still different from the identification problem addressed in this paper, particularly given the differences in the models being identified. The identification strategy in this paper does not restrict to auctions or independent values.

Even with the discrete actions allowed in this paper, the models considered in this paper are distinct from the models considered in the literature on the "econometrics of (entry) games" (e.g., Tamer (2003), Aradillas-López and Tamer (2008), Ciliberto and Tamer (2009), Aradillas-López (2010), Bajari, Hong, Krainer, and Nekipelov (2010), Bajari, Hong, and Ryan (2010), de Paula and Tang (2012), Kline and Tamer (2012, 2016), de Paula (2013), Kline (2015, 2016), Aradillas-López (2020), Ciliberto, Murry, and Tamer (2021)). Simply put, a setting involving allocations and transfers is different from a setting of market entry, and so the models and corresponding identification strategies are also different. Nevertheless, the results do apply to some models of strategic (non-"price taking") market behavior, which can describe behavior after entry into a market, as in Example 6.

The remainder of the paper is organized as follows. Section 2 sets up the allocation-transfer game framework studied in this paper. Section 3 provides the identification strategy. Finally, Section 4 concludes. The appendices collect a variety of technical details and extensions. Appendix A provides sufficient conditions for point identification, relating to a discussion in Section 3.7 about the "limit" when the number of actions either is or increases to become an interval. Appendix B provides examples of the allocation-transfer games framework studied in this paper. Appendix C collects the proofs. Appendix D shows that identification of some features of the distribution of valuations is robust to partial failures of the equilibrium assumption. Appendix E provides sufficient conditions for a particular assumption used in Appendix A.

2. Allocation-transfer game framework

There are $N \geq 2$ players in the game, which determines the allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions of the players. Players are indexed by $i = 1, 2, \ldots, N$. In principle, the results could apply to some "single-player games" with $N = 1$, if the assumptions hold in such a game, but the focus is on multiple-player games.

2.1. **Utility functions.** Player i has valuation θ_i for a unit of the object. The utility of player i with valuation θ_i, and who receives allocation x_i of the object and transfers away ("pays") t_i units of money is

$$U(\theta_i, x_i, t_i) \equiv \theta_i x_i - t_i.$$

The sign of t_i is unrestricted, so player i can be "paid" if t_i is negative. The allocation and transfers are determined by the game, described shortly in Section 2.3. For example, the monetary transfer could be the payment in an auction model, the "price" in a market model, or the contribution in a public good provision model. This utility function is standard in the economic theory literature.

It is common knowledge among the players that the valuations $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_N)$ are drawn from the joint distribution $F(\theta)$. The actual realization θ_i is the private information of player i.

Assumption 1 (Dependent valuations)**.** *It is common knowledge among the players that θ is drawn from $F(\theta)$, and θ_i is the private information of player i.*

The econometrician need not know the support of θ. It is allowed that θ is continuous, discrete, or some combination. The identification results allow dependent valuations. In particular, allowing dependent valuations allows the realistic possibility that a particular player can draw inferences about the valuations of the other players, on the basis of its own private valuation. The identification results do simplify under the further assumption of independent valuations:

Assumption 2 (Independent valuations)**.** *In addition to Assumption 1, player valuations are independent, in the sense that the components of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ are independent random variables, so $F(\theta) = F_1(\theta_1)F_2(\theta_2)\cdots F_N(\theta_N)$.*

Throughout the paper, the case of independent valuations is treated as an important special case, alongside the more general results allowing dependent valuations. Even under the assumption of independent valuations, it is not assumed that players necessarily draw their valuation from the same distribution, so $F_i(\cdot)$ need not equal $F_j(\cdot)$, which is useful for example to model "weak" and "strong" bidders in auctions or asymmetries between buyers and sellers in models of market behavior. Symmetry is allowed as a special case.

2.2. **Actions.** After realizing θ_i, player i takes an action a_i from its action space $\mathcal{A}_i$. The interpretation of the actions depends on the game, and includes efforts in contest models, bids in auction models, announcements (bids/asks) in market models, and contributions in public good provision models.

Assumption 3 (Action space is ordered)**.** *For each $i \in \{1, 2, \ldots, N\}$, the econometrician knows the action space for player i is $\mathcal{A}_i \subseteq \mathbb{R}$.*

This allows the action space to be continuous, discrete, or some combination of continuous and discrete. Particularly for any discrete part of the action space, there is not necessarily a "numerical interpretation" of the actions in $\mathcal{A}_i$, similar to how the numerical encodings of the categories in categorical choice models may or may not actually have a substantive "numerical interpretation." For example, in games with voluntary participation including auctions with participation costs, one of the actions is the "do not participate" action. The action "DNP" in such games would have a special ("non-numerical") interpretation of "do not participate (in the auction)."

Even if there is no "numerical interpretation" of the actions in $\mathcal{A}_i$, it is important that the action space is ordered because it is assumed that players use monotone strategies, which requires by definition that the action space is ordered. The numerical encoding of "special" actions as numbers in $\mathcal{A}_i$ respects the ordering of the actions. For example, in auctions with voluntary participation, generically players with low valuations choose to not participate, so it makes sense to define DNP to be the lowest possible action, in order for the equilibrium strategy to be monotone. It could be that DNP is encoded as -1 or -2, for example. The specific numerical encoding is irrelevant.

2.3. **Allocations and transfers.** The vector of all players' actions is $a = (a_1, a_2, \ldots, a_N)$, the vector of all players' allocations is $x = (x_1, x_2, \ldots, x_N)$, and the vector of all players' transfers is $t = (t_1, t_2, \ldots, t_N)$.

The game determines the allocations and transfers on the basis of the actions taken by the players. Even for a given profile of actions, non-deterministic allocations and transfers are allowed, for example to allow "noise" in the process of determining a winner in a contest (see Example 1). On the basis of all players' actions a, the realized allocation and transfer is a realization[2] from the joint distribution of

$$(\widetilde{x}(a), \widetilde{t}(a)) = (\widetilde{x}_1(a), \widetilde{x}_2(a), \ldots, \widetilde{x}_N(a), \widetilde{t}_1(a), \widetilde{t}_2(a), \ldots, \widetilde{t}_N(a)),$$

where $\widetilde{x}_i(a)$ (resp., $\widetilde{t}_i(a)$) is a random variable that characterizes the distribution of allocations (resp., transfers) for player i given that the players take actions a. These distributions characterizing the allocations and transfers are part of the specification of the game rules.

[2]By construction, these realizations are draws from the joint distribution and therefore by construction are independent from all other model quantities (e.g., the valuations of the players). This condition formalizes the notion that the allocation and transfer "don't depend on" anything except the actions of the players, and is (often implicitly) a standard condition in the related economic theory literature. Of course, the realized allocation and transfer will *indirectly* depend on the players' valuations, since the players' valuations determine the players' actions and the players' actions determine the realized allocation and transfer. For example, in the case of a tie for high bid in an auction, the auctioneer could flip a coin to determine who wins, but the outcome of the coin flip cannot somehow be "correlated" with the valuations of the players.

The variable x_i (resp., t_i) is player i's realized allocation (resp., transfer) in its utility function. If $(\widetilde{x}_1(a), \widetilde{x}_2(a), \ldots, \widetilde{x}_N(a), \widetilde{t}_1(a), \widetilde{t}_2(a), \ldots, \widetilde{t}_N(a))$ is a degenerate random variable, then the allocation and transfer is deterministic when the players take actions a. As a function of all players' actions, the *expected* allocation to player i is $\overline{x}_i(a) = E(\widetilde{x}_i(a))$ and the *expected* transfer from player i is $\overline{t}_i(a) = E(\widetilde{t}_i(a))$.

Although not assumed for the identification results, in most instances of allocation-transfer games, the allocation to and transfer from player i are weakly increasing functions of a_i. The allocation to player i can be either a weakly increasing function of a_{-i} (e.g., public good provision in Example 5) or weakly decreasing function of a_{-i} (e.g., most auctions in Example 2).

Per the standard assumption from the economic theory literature that the game is common knowledge among the players, the players know the distributions of $(\widetilde{x}(\cdot), \widetilde{t}(\cdot))$. In other words, the players know the "rules" of the game.

The identification results can apply regardless of whether or not the econometrician knows the distributions of $(\widetilde{x}(\cdot), \widetilde{t}(\cdot))$, and/or the *expected* allocations and transfers $(\overline{x}(\cdot), \overline{t}(\cdot))$. In particular, any "randomness" that underlies non-deterministic allocations and transfers need not be explicitly modeled or known by the econometrician. Intuitively, if the econometrician does not know these objects, then it is possible to use the data to identify these objects.

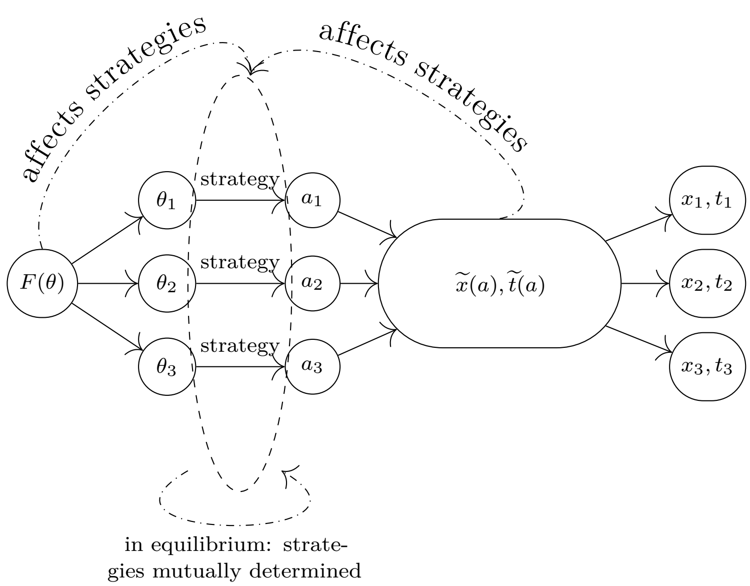

FIGURE 1. Graphical summary of game in the case of $N = 3$.

2.4. **Diagram of game framework.** Figure 1 provides a sketch of the basic idea of the allocation-transfer game framework studied in this paper. The game determines the allocations and transfers (the x and t variables) on the basis of the actions of the players (the a variables). The strategy of player i determines the action a_i taken by player i as a function of the realized valuation θ_i of player i. The strategies depend implicitly on the rules of the game. In equilibrium, the strategies also depend on the strategies used by the other players, in the sense of mutual best responses. Obviously, as illustrated via specific examples in Appendix B, many economic environments can be modeled using this allocation-transfer game framework. This includes contests, auctions, procurement auctions and related models of oligopoly competition, partnership dissolution, public good provision, and strategic (non-"price taking") market behavior. The results apply to this class of allocation-transfer games, and therefore do not rely on specifics of particular examples. The range of examples in Appendix B shows the generality of this allocation-transfer game framework.

2.5. **Data and identification problem.** The identification problem concerns recovering the distribution of valuations from observing many instances ("plays") of the game. For context, the related literature on identification in auctions has typically considered this identification problem in the case of auctions specifically. Variables relating to the actions, allocations, and transfers in *upper-case letters* represent quantities in the data, whereas quantities in *lower-case letters* represent variables in the underlying game. For example, A_i is the realized action in the data from player i, whereas a_i is the action variable in the underlying game from player i. Therefore, from each play of the game, the realized actions are $A = (A_1, A_2, \dots, A_N)$, the realized allocations are $X = (X_1, X_2, \dots, X_N)$, and the realized transfers are $T = (T_1, T_2, \dots, T_N)$. Unless otherwise stated, the econometrician observes population data on the actions, allocations, and transfers. Hence, unless otherwise stated, the population data is $P(A, X, T)$.

The realized allocations and realized transfers are linked to the realized actions through the game: in each instance of the game, by definition (X, T) is a draw from

$$(\widetilde{x}(A), \widetilde{t}(A)) = (\widetilde{x}_1(A), \widetilde{x}_2(A), \dots, \widetilde{x}_N(A), \widetilde{t}_1(A), \widetilde{t}_2(A), \dots, \widetilde{t}_N(A)),$$

the possibly non-deterministic allocation and transfer distributions given action profile A of the players. In the case of deterministic allocation and deterministic transfer, for a particular action profile A, then it can be understood that simply $X = \widetilde{x}(A) = (\widetilde{x}_1(A), \widetilde{x}_2(A), \dots, \widetilde{x}_N(A))$ and $T = \widetilde{t}(A) = (\widetilde{t}_1(A), \widetilde{t}_2(A), \dots, \widetilde{t}_N(A))$.

In some cases, the identification strategy can be based on less than full data on $P(A, X, T)$. Specifically, if the econometrician specifies a complete model of the game, in the sense that the econometrician knows $(\widetilde{x}(a), \widetilde{t}(a))$, then the identification strategy can be based on only $P(A)$. If the game involves a "two-part transfer," as in an auction with a participation cost, then the identification strategy can in certain cases be based on data from only one part of the transfer. See the discussion in Section 3.4.

3. Identification analysis

3.1. **Baseline assumptions.** The following baseline assumptions are used. These assumptions are standard from the economic theory literature.

The players are assumed to be risk neutral, and therefore the *expected* allocations and transfers $\overline{x}_i(a)$ and $\overline{t}_i(a)$ determine *ex post* expected utility of player i as a function of its valuation and all players' actions:

$$\overline{U}_i(\theta_i, a) = \theta_i \overline{x}_i(a) - \overline{t}_i(a).$$

In this paper, *ex post* expected utility refers to after the realization of the actions of all players in the game, which still can involve the expectation with respect to the non-degenerate randomness of the allocation rule and transfer rule. Because of expected utility, the utility that is actually realized (based on actually realized allocation and transfer) plays no role distinct from *ex post* expected utility. *Ex interim* expected utility refers to before the realization of the actions of all players in the game, but after an individual player realizes its own valuation, which involves taking the expectation with respect to the player's beliefs about the other players' actions and the randomness of the allocation rule and transfer rule.

Because player i does not know the actions of the other players when it chooses its action, it must form beliefs about the actions of the other players. With dependent valuations, the beliefs held by player i about the actions of the other players depends on player i's realized valuation, so player i's beliefs are a distribution $\Pi_i(a_{-i}|\theta_i)$, defined over the actions of the other players, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, that conditions on player i's realized valuation θ_i. In other words, with dependent valuations, players might be able to draw inferences about other players' valuations, and therefore other players' actions.

Independent valuations *Under Assumption 2 (Independent valuations), player i's beliefs are $\Pi_i(a_{-i})$, independent of player i's realized valuation. That is because with independent valuations, the realized valuation of player i does not revise the beliefs of player i about θ_{-i}, and therefore does not revise the beliefs of player i about a_{-i}.* ★

Therefore, *ex interim* expected utility of player i as a function of its valuation and its action is

$$V_i(\theta_i, a_i) = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i).$$

With independent valuations, θ_i affects player i's *ex interim* expected utility only through the direct effect on the value of the object. With dependent valuations, θ_i also affects the expected allocation and expected transfer experienced by player i, even for a fixed action a_i, since player i's expected allocation and expected transfer depend on player i's beliefs about the other players' actions, and therefore on θ_i. This substantially complicates the identification problem under dependent valuations, compared to independent valuations.

Given this *ex interim* expected utility function, player i rationally takes an action that maximizes its *ex interim* expected utility given its realized valuation, so that its strategy $a_i(\theta_i)$ is supported on the set of actions that maximizes *ex interim* expected utility:

$$a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i)). \quad (1)$$

The identification strategy is based around the utility maximization problem in Equation 1 facing each player as a function of its realized valuation.

Assumption 4 (Optimal strategy is used)**.** *For each $i \in \{1, 2, \ldots, N\}$, for each possible valuation θ_i, player i uses a strategy $a_i(\theta_i)$ when it has valuation θ_i, with $a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i))$, so each action taken according to the strategy $a_i(\theta_i)$ maximizes* ex interim *expected utility.*

In this assumption and other places, "possible valuation" means a valuation that is possible according to the (unknown) distribution of valuations. This assumption means that player i is rational, in the sense that it uses a strategy that maximizes its utility given its beliefs. Assumption 4 does not state that player i has correct beliefs. Instead, the subsequent Assumption 5 states that player i has correct beliefs. Also, Assumption 4 allows the use of a mixed strategy, but the identification strategy is based on the assumption of monotone equilibrium in monotone pure strategies, as formalized and discussed subsequently in Assumption 7. Breaking up the assumptions makes it easier to explain the identification strategy, by making it easier to refer to separate roles of the assumptions of using an optimal strategy, correct beliefs, and monotone equilibrium.

Let $P(A, X, T, \theta)$ be the "infeasible" data, regardless of whether those variables are observed by the econometrician. Then let $P(A_{-i}|\theta_i)$ be the realized distribution in the "infeasible" data over $A_{-i} = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_N)$ conditional on the realized valuation θ_i of player i. Of course, θ_i is not observed by the econometrician, so the econometrician cannot condition on θ_i. In a Bayes Nash equilibrium, each player's beliefs are correct and correspond to the actual distribution of actions of the other players, in the sense that, for each player i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B. In other words, the beliefs of player i about a_{-i} when player i has valuation θ_i is equal to the actual realized distribution of A_{-i} when player i has valuation θ_i. This is the standard definition of correct beliefs with incomplete information.

Assumption 5 (Correct beliefs)**.** *For each $i \in \{1, 2, \dots, N\}$, player i has correct beliefs, in the sense that, for each possible valuation θ_i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B.*

Independent valuations *Under Assumption 2 (Independent valuations), the assumption of correct beliefs is $\Pi_i(a_{-i} \in B) = P(A_{-i} \in B)$, since then beliefs do not depend on θ_i.* ★

As in other incomplete information setups, this assumption of correct beliefs implicitly supposes the realized distribution of actions (i.e., the data) comes from a single equilibrium corresponding to the players' beliefs. If multiple equilibria were played in the data, even with "correct beliefs" in each equilibrium, the realized distribution over actions in the data would be a mixture over the beliefs held by the player across equilibria, and thus the realized distribution over actions in the data would not equal players' beliefs. However, the econometrician need not have any *ex ante* knowledge of *which* equilibrium is selected in the case of multiple equilibria. If there is a unique equilibrium of the game, then obviously the assumption that the data comes from a single equilibrium is automatically satisfied. Furthermore, the economic theory literature has many results on equilibrium uniqueness, sometimes under further assumptions on the "nature" of the equilibrium, for instance with some results that show cases where there is a unique equilibrium that involves using monotone strategies, even if there are other equilibria that do not involve monotone strategies. In that way, assuming the use of monotone strategies "helps" with the assumption that the data comes from a single equilibrium, by ruling out certain equilibria. The combination of Assumptions 4 and 5 entails a Bayesian Nash equilibrium.

Under correct beliefs held by player i, $V_i(\theta_i, a_i) = \theta_i E_P(\overline{x}_i(a_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i, A_{-i})|\theta_i)$.

Appendix D shows that identification of some features of the distribution of valuations is robust to partial failures of these assumptions that all players use an optimal strategy with correct beliefs, basically as long as some players (but not necessarily all players) satisfy these assumptions.

Assumption 6 (Counterfactual *ex interim* expected utility maximization problem has a solution)**.** *For each $i \in \{1, 2, \ldots, N\}$, $\max_{a_i \in \mathcal{A}_i} (\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i''))$ has a solution for any possible valuations θ_i and θ_i''.*

Assumption 4 (Optimal strategy is used) states that the *ex interim* expected utility maximization problem has a solution when $\theta_i = \theta_i''$. Standard conditions imply that a solution exists even when $\theta_i \neq \theta_i''$. In particular, Assumption 6 holds trivially when $|\mathcal{A}_i|$ is finite. Also, clearly Assumption 6 is redundant under Assumption 2 (Independent valuations) given Assumption 4 (Optimal strategy is used).

3.2. **Monotone equilibrium.** The main assumption of the identification strategy is monotone equilibrium.

Assumption 7 (Weakly increasing strategy is used)**.** *For each $i \in \{1, 2, \ldots, N\}$, for each possible valuation θ_i, $a_i(\theta_i)$ is a pure strategy. And, for each $i \in \{1, 2, \ldots, N\}$, $a_i(\cdot)$ is a weakly increasing*[3] *function.*

The use of pure strategies implies that $a_i(\theta_i)$ is a particular action (i.e., a pure strategy) rather than a non-degenerate distribution (i.e., a mixed strategy). Equilibrium existence in pure strategies is a general result for games with incomplete information. The economic theory (and existence) of such equilibria in pure strategies has been studied, for example, in Milgrom and Weber (1982, 1985), Dasgupta and Maskin (1986), Plum (1992), Reny (1999), Lizzeri and Persico (2000), Maskin and Riley (2003), and Jackson and Swinkels (2005) in addition to citations elsewhere in this paper, particularly Appendix B, among a huge literature.

[3]It is straightforward to accommodate a weakly *decreasing* strategy, because a weakly *decreasing* strategy can be translated into a weakly *increasing* strategy by flipping the signs on the allocation rule and valuations, because if the strategy is weakly *decreasing* in the valuation θ_i, then the strategy is weakly *increasing* in the "negative valuation" $\hat{\theta}_i = -\theta_i$ with "negative allocation" $\hat{x}_i(a) = -\widetilde{x}_i(a)$. Note that $\hat{\theta}_i \hat{x}_i(a) = \theta_i \widetilde{x}_i(a)$ so utility is unaffected by flipping the signs in this way.

The assumption of monotone equilibrium is intuitive. For example, in applications to contests (Example 1), a monotone strategy simply requires the intuitive condition that players put forth effort as a weakly increasing function of their valuation for the object awarded by the contest. Or for another example, in applications to auctions (Example 2), a monotone strategy simply requires the intuitive condition that players make bids that are weakly increasing functions of their valuation for the object being auctioned. Appendix B provides a variety of other examples of games for which Assumption 7 is intuitive.

The economic theory literature has emphasized the importance of proving existence of equilibrium in monotone strategies. General results establishing conditions for existence of pure strategy equilibria in monotone strategies include Maskin and Riley (2000), Athey (2001), McAdams (2003, 2006), and Reny (2011). Such results establish general conditions on the game that are sufficient for existence of monotone equilibrium. Moreover, again as cited elsewhere in this paper, particularly Appendix B, the economic theory literature has also established existence of pure strategy equilibria in monotone strategies in the context of specific games. Many economic theory papers establishing Assumption 7 assume affiliated valuations. Particularly in the context of affiliation in auctions, see Milgrom (2004, Section 5.4.1) for details. Further, many papers on identification in auctions assume affiliated valuations. The identification strategy in this paper does not require affiliation, as long as Assumption 7 is satisfied. Equilibria in monotone strategies can exist even without affiliated valuations, see for example Monteiro and Moreira (2006).

This paper uses monotonicity in a fundamentally different way from other common uses of monotonicity in econometrics. In other areas of econometrics, monotonicity commonly relates to the functional relationship between two observed variables, and the functional relationship is the object of interest. Monotonicity assumptions are commonly used in regression models or treatment effects models that relate an outcome to a treatment. Monotonicity has been imposed as a shape restriction on the estimator in regression models (e.g., Mukerjee (1988), Ramsay (1988, 1998), and Mammen (1991)), and has been used in the identification of treatment effects models (e.g, Manski (1997), and Manski and Pepper (2000, 2009)). By contrast, when assuming use of monotone strategies, the monotonicity relates to the equilibrium functional relationship between the observed action and the unobserved valuation, and the distribution of the unobserved valuations is the object of interest.

Under Assumption 7, players use *weakly* increasing strategies, which accommodates the possibility that player i with valuation θ_i takes the action $a_i(\theta_i)$ and player i with valuation $\theta_i' \neq \theta_i$ also takes the same action $a_i(\theta_i') = a_i(\theta_i)$. Such "flat spots" in the strategy would necessarily arise due to pooling when there is discreteness of the action space. Such "flat spots" can also arise even without discreteness in the action space, for example as discussed in Example 6. Under Assumption 7, the set $\{\theta_i : a_i(\theta_i) = a_i^*\}$ of valuations θ_i that use given action $a_i^* \in \mathcal{A}_i$ is necessarily an interval, which can be the empty set, a singleton set, or a non-degenerate interval (possibly infinite, and possibly including or not the endpoints).[4]

The identification problem caused by the possibility of "flat spots" in the strategies is exacerbated by the fact that the identification strategy accommodates dependent valuations. Specifically, the beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

Assumption 8 (Monotone effect of counterfactual beliefs on utility)**.** *For each $i \in \{1, 2, \ldots, N\}$, and any possible valuations θ_i and θ_i'', there is some selection*

$$a_i(\theta_i; \theta_i'') \in \arg\max_{a_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i'')\right)$$

with $a_i(\theta_i; \theta_i) = a_i(\theta_i)$ from Assumption 7, such that when $\theta_i' \leq \theta_i''$, with either $\theta_i' = \theta_i$ or $\theta_i'' = \theta_i$,

$$\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i') \geq$$
$$\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i'').$$

The action $a_i(\theta_i; \theta_i'')$ maximizes the "counterfactual" *ex interim* expected utility of player i with valuation θ_i and "counterfactually" the beliefs of a player with valuation θ_i'' possibly not equal to θ_i. Assumption 8 states that the "counterfactual" *ex interim* expected utility experienced by player i that has valuation θ_i that uses such an action $a_i(\theta_i; \theta_i'')$ and "counterfactually" has the beliefs of valuation θ_i' with $\theta_i' \leq \theta_i''$ is weakly greater than that under the beliefs with valuation θ_i''. A sufficient condition is that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i; \theta_i''), a_{-i})|\theta_i')$ is a weakly decreasing function of θ_i'.

[4] Suppose $\{\theta_i : a_i(\theta_i) = a_i^*\}$ is not the empty set. And suppose that $a_i(\theta_i) = a_i^*$ and $a_i(\theta_i') = a_i^*$. Suppose without loss of generality that $\theta_i \leq \theta_i'$. Since $a_i(\cdot)$ is weakly increasing, any valuation between θ_i and θ_i' also uses action a_i^*.

Hence, Assumption 8 can be interpreted as stating that utility is monotone in the "counterfactual beliefs" arising from "counterfactual" valuations.

If valuations are independent, then beliefs do not depend on valuation, so Assumption 8 is satisfied. Further, even when valuations are dependent, Assumption 8 is satisfied when valuations are suitably "positively dependent" (i.e., affiliated as in Milgrom (2004, Section 5.4.1), or alternatively, with the distribution of $\theta_{-i}|\theta_i$ monotonic in θ_i in the usual multivariate stochastic order), all players have correct beliefs (per Assumption 5) and use weakly increasing strategies (per Assumption 7), and *ex post* utility $\theta_i \overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i}) - \overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})$ of player i weakly decreases with the actions of the other players, when player i takes the action $a_i(\theta_i;\theta_i'')$.

Lemma 1 (Sufficient conditions for Assumption 8)**.** *Suppose that for each $i \in \{1, 2, \ldots, N\}$, and any possible valuations θ_i and θ_i'', there is some selection*

$$a_i(\theta_i;\theta_i'') \in \arg\max_{a_i \in \mathcal{A}_i} (\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i''))$$

with $a_i(\theta_i;\theta_i) = a_i(\theta_i)$ from Assumption 7, such that $\theta_i \overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i}) - \overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})$ is a weakly decreasing function of a_{-i}. Suppose Assumptions 5 (Correct beliefs) and 7 (Weakly increasing strategy is used) are satisfied. Suppose either: (a) valuations are affiliated, or (b) the distribution of $\theta_{-i}|(\theta_i = \theta_i')$ is stochastically smaller than the distribution of $\theta_{-i}|(\theta_i = \theta_i'')$ in the usual multivariate stochastic order, when $\theta_i' \leq \theta_i''$. Then Assumption 8 is satisfied.

Under the conditions of Lemma 1, if player i "counterfactually" has the beliefs of player i with valuation θ_i' with $\theta_i' \leq \theta_i''$ then player i believes the other players to take weakly lower actions compared to the case of having the beliefs of player i with valuation θ_i'', and therefore *ex interim* expected utility is weakly greater under "counterfactual" beliefs θ_i' compared to "counterfactual" beliefs θ_i'' since *ex post* utility is weakly greater when the actions of the other players are weakly lower. In other words, Assumption 8 is the idea that player i believes itself to be "better off" when it has beliefs that involve "weaker" opponents. The conditions in Lemma 1 are sufficient but not necessary for Assumption 8, so a violation of these conditions does not imply that Assumption 8 fails. In particular, as noted above, Assumption 8 is satisfied with independent valuations, regardless of any other condition. Also, if the direction of the monotonicity happens to be opposite that of Assumption 8, it is straightforward to adjust the identification result accordingly (essentially the inequality $z_i' < a_i < z_i''$ switches directions in the statement of Theorem 1).

3.3. **Definitions of stochastic ordering.** The identification strategy results in bounds on the multivariate distribution of valuations in terms of the usual multivariate stochastic order, which concerns both the marginal distributions of each player's valuation and the dependence structure ("correlation") of the valuations.

Definition 1 (Upper set)**.** Let $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$. A set $U \subseteq \mathbb{R}^d$ is an upper set if $x \in U$ and $y \geq x$ implies that $y \in U$. Per the standard, the condition $y \geq x$ is equivalent to $y_j \geq x_j$ for all $j = 1, 2, \ldots, d$.

Definition 2 (Usual multivariate stochastic order)**.** Let A and B be d-dimensional random vectors, with probability laws P_A and P_B. A is stochastically larger than B in the usual multivariate stochastic order if $P_A(U) \geq P_B(U)$ for all Borel measurable upper sets $U \subseteq \mathbb{R}^d$. And A is stochastically smaller than B in the usual multivariate stochastic order if B is stochastically larger than A in the usual multivariate stochastic order.

As formalized in Shaked and Shanthikumar (2007, Theorem 6.B.1), A is stochastically larger than B in the usual multivariate stochastic order exactly when there are $\hat{A}$ and $\hat{B}$ defined on the same probability space, such that $\hat{A}$ has the same distribution as A and $\hat{B}$ has the same distribution as B, and such that $\hat{A} \geq \hat{B}$ with probability 1. In the usual multivariate stochastic order, the partial identification result establishes that the random vector of valuations $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is stochastically larger than a certain random vector (i.e., "the distribution of θ is bounded below") and is stochastically smaller than another certain random vector (i.e., "the distribution of θ is bounded above"). The random vectors that are the upper and lower bounds for θ are themselves identified quantities, and have a constructive definition as a function of the observable data.

As discussed in Shaked and Shanthikumar (2007, Chapter 6), by the standard properties of the usual multivariate stochastic order, the partial identification result in terms of the usual multivariate stochastic order also implies partial identification of other quantities, including expectations of functions of the valuations and the multivariate cumulative distribution function of the valuations. In particular, the condition that the random vector A is stochastically larger than the random vector B in the usual multivariate stochastic order is equivalent to the condition that $E(\phi(A)) \geq E(\phi(B))$ for all weakly increasing functions ϕ for which the expectations exist.

In particular, because $\phi(X) = 1[X \leq t]$ is weakly decreasing in X, the condition that A with distribution function F_A is stochastically larger than B with distribution function F_B in the usual multivariate stochastic order implies that $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$.

As formalized in Definition 3, the condition that $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$ is known as the lower orthant order (e.g., Shaked and Shanthikumar (2007, Chapter 6.G.1)). The lower orthant order is a distinct sense of stochastic ordering. For random vectors, unlike for scalar random variables, the lower orthant ordering is implied by, but does not imply, the usual multivariate stochastic ordering. See Müller (2001) for more about the relationships between the senses of stochastic ordering when A and B are multivariate normal.

Definition 3 (Lower orthant stochastic order)**.** Let A and B be d-dimensional random vectors, with cumulative distribution functions F_A and F_B. A is stochastically larger than B in the lower orthant stochastic order if $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$. And A is stochastically smaller than B in the lower orthant stochastic order if B is stochastically larger than A in the lower orthant stochastic order.

Bounds on the distribution of valuations in the usual multivariate stochastic order also imply bounds on other quantities derived from the distribution of valuations, as discussed in Shaked and Shanthikumar (2007, Chapter 6). In their independent private values English auction setup, Haile and Tamer (2003) have shown how to use lower orthant bounds on the scalar distribution of valuations to bound the optimal reserve price in auctions.

3.4. **Game-structure identification of differences.** Let $\mathcal{A}_i^d$ be the support of A_i, the actions taken in the data by player i. Potentially, $\mathcal{A}_i^d$ is a proper subset of $\mathcal{A}_i$. Identification of θ_i depends on identification of aspects of the structure of the game itself, as follows:

Definition 4 (Specification with game-structure identification of differences)**.** A specification $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i)^4$ of player i with $z_i', z_i'' \in \mathcal{A}_i^d$, is a specification with game-structure identification of differences if

$$E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') \text{ and } E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$$

are point identified. The set of specifications with game-structure identification of differences is $\mathcal{R}_i$.

Definition 4 involves expected values of the allocation rule and transfer rule with respect to the observed distribution of $P(A)$. Therefore, one sufficient condition for game-structure identification of differences at any given specification (a_i, z_i, z_i', z_i'') is for the allocation rule and transfer rule to be known *ex ante* by the econometrician. Thus, game-structure identification of differences can fail only when the econometrician does not *ex ante* know the allocation rule and/or transfer rule.

In fact, for particular functional forms of the allocation rule and transfer rule, it suffices for the econometrician to know less than the entire rule, since only differences are relevant for Definition 4. For instance, this can accommodate an unknown (to the econometrician) participation cost.[5]

Even if the allocation rule and transfer rule is not known by the econometrician, $\overline{x}_i(a_i, a_{-i}) = E_P(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a_i, a_{-i}) = E_P(T_i|A_i = a_i, A_{-i} = a_{-i})$ are point identified quantities under standard conditions on identification/estimation of conditional expectations. In the interest of space, just one such sufficient condition for game-structure identification of differences is reported here. Let $\mathcal{A}^d$ be the support of the observed actions $(A_1, A_2, \ldots, A_N)$.

Lemma 2 (Sufficient conditions for game-structure identification of differences with discrete actions and unknown allocation rule and/or transfer rule)**.** *Suppose that Assumptions 1 (Dependent valuations) and 3 (Action space is ordered) are satisfied. Suppose the data is $P(A, X, T)$. If $\mathcal{A}_i$ is discrete for all players i, and $\mathcal{A}^d = \prod_i \mathcal{A}_i^d$, then any specification of actions $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i^d)^4$ of player i is a specification with game-structure identification of differences per Definition 4. Consequently, $\mathcal{A}_i^d \times \mathcal{A}_i^d \times \mathcal{A}_i^d \times \mathcal{A}_i^d \subseteq \mathcal{R}_i$.*

Independent valuations *Under Assumption 2 (Independent valuations), A_{-i} is independent of A_i and therefore z_i' and z_i'' effectively play no role in Definition 4. So, under Assumption 2, a specification $(a_i, z_i) \in (\mathcal{A}_i)^2$ is a specification with game-structure identification of differences if it satisfies the condition in Definition 4, without the conditioning on z_i' and z_i''. Hence, under Assumption 2, the dimension of elements of $\mathcal{R}_i$ changes.* ★

3.5. **Identification results.** As often with partial identification results, the identified set can depend on *ex ante* known bounds on the partially identified quantity.

[5] In an auction, for example, a participation cost is a transfer paid by any bidder who places a bid (rather than taking the "do not participate" action). Suppose that $\overline{t}_i(a_i, a_{-i}) \equiv \overline{t}_{i1}(a_i, a_{-i}) + \overline{t}_{i2}(a_i, a_{-i}) \equiv \overline{t}_{i1}(a_i, a_{-i}) + \overline{t}_{i2}(a_i)$, so that the transfer is the sum of two transfers, one of which depends only on a_i. Then the relevant difference is $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'') = \big(E_P(\overline{t}_{i1}(a_i, A_{-i})|A_i = z_i') + \overline{t}_{i2}(a_i)\big) - \big(E_P(\overline{t}_{i1}(z_i, A_{-i})|A_i = z_i'') + \overline{t}_{i2}(z_i)\big)$. It would therefore suffice for the econometrician to know $\overline{t}_{i1}(a_i, a_{-i})$ for all (a_i, a_{-i}) and $\overline{t}_{i2}(a_i) - \overline{t}_{i2}(z_i)$ at least for the specified (a_i, z_i). If $\overline{t}_{i2}$ is the participation cost, and the cost of participating is the same for actions a_i and z_i, then the econometrician knows that $\overline{t}_{i2}(a_i) - \overline{t}_{i2}(z_i) = 0$ even if the econometrician does not know the participation cost.

Assumption 9 (Known bounds on valuations)**.** *For each $i \in \{1, 2, \ldots, N\}$, the valuation θ_i must be in the set $[\Theta_{Li}, \Theta_{Ui}]$.*

By setting $\Theta_{Li} = -\infty$ and $\Theta_{Ui} = \infty$, it is possible to check the identification result without such known bounds. In many games, it might be reasonable to set $\Theta_{Li} = 0$, reflecting that the object has non-negative value to all players. Assumption 9 is not the statement that the support of the valuations is $[\Theta_{Li}, \Theta_{Ui}]$, but rather is the statement that the support of the valuations is contained within $[\Theta_{Li}, \Theta_{Ui}]$. Hence, as also stated in Assumption 1 (Dependent valuations), the econometrician need not know the support of the valuations.

In order to state the identification result, let

(2)
$$\Phi_{Li}(a_i) = \max \begin{cases} \sup \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} : \\ \qquad z_i' < a_i < z_i'', \\ \qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\}, \\ \qquad (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \\ \Theta_{Li} \end{cases}$$

and

(3)
$$\Phi_{Ui}(a_i) = \min \begin{cases} \inf \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} : \\ \qquad z_i' < a_i < z_i'', \\ \qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}, \\ \qquad (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \\ \Theta_{Ui} \end{cases}$$

where Θ_{Li} and Θ_{Ui} are *ex ante* known bounds on valuations from Assumption 9.

Let

$$\Upsilon_{Li}(a_i) = \sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i') \text{ and } \Upsilon_{Ui}(a_i) = \inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i'). \tag{4}$$

This provides the notation for the following main partial identification result. Section 3.6 contains a sketch of the identification strategy.

Theorem 1. *Under Assumptions 1 (Dependent valuations), 3 (Action space is ordered), 4 (Optimal strategy is used), 5 (Correct beliefs), 6 (Counterfactual* ex interim *expected utility maximization problem has a solution), 7 (Weakly increasing strategy is used), 8 (Monotone effect of counterfactual beliefs on utility), and 9 (Known bounds on valuations), the distribution of valuations θ is partially identified, and the identification is constructive, because the distribution of θ is stochastically larger than the distribution of $(\Upsilon_{L1}(A_1), \Upsilon_{L2}(A_2), \ldots, \Upsilon_{LN}(A_N))$ and is stochastically smaller than the distribution of $(\Upsilon_{U1}(A_1), \Upsilon_{U2}(A_2), \ldots, \Upsilon_{UN}(A_N))$, in the sense of the usual multivariate stochastic order, where $(A_1, A_2, \ldots, A_N)$ is distributed according to the data $P(A, X, T)$ and $\Upsilon_{Li}(\cdot)$ and $\Upsilon_{Ui}(\cdot)$ are the identifiable functions given in Equation 4.*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations), drop Assumption 6 (Counterfactual* ex interim *expected utility maximization problem has a solution), and 8 (Monotone effect of counterfactual beliefs on utility), and replace the Υ functions with the Γ functions defined in Equation 7, below.*

Let

$$\Xi_{Li}(a_i) = \max \begin{cases} \sup \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})) - E_P(\bar{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))} : \\ \qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) > 0\}, \\ \qquad (a_i, z_i) \in \mathcal{R}_i \end{cases} \\ \Theta_{Li} \end{cases} \tag{5}$$

and

$$\Xi_{Ui}(a_i) = \min \begin{cases} \inf \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})) - E_P(\bar{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))} : \\ \qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) < 0\}, \\ \qquad (a_i, z_i) \in \mathcal{R}_i \end{cases} \\ \Theta_{Ui} \end{cases} \tag{6}$$

Let

$$\Gamma_{Li}(a_i) = \sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Li}(a_i') \text{ and } \Gamma_{Ui}(a_i) = \inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Ui}(a_i'). \tag{7}$$

Further, under Assumption 2 (Independent valuations), game-structure identification of differences can be established: $E_P(\bar{t}_i(a_i, A_{-i})) = E_P(T_i|A_i = a_i)$, $E_P(\bar{x}_i(a_i, A_{-i})) = E_P(X_i|A_i = a_i)$, $E_P(\bar{t}_i(z_i, A_{-i})) = E_P(T_i|A_i = z_i)$, *and* $E_P(\bar{x}_i(z_i, A_{-i})) = E_P(X_i|A_i = z_i)$. ★

Under Assumption 2 (Independent valuations), the identification result in Theorem 1 is "nearly sharp" in the sense that the lower bound and upper bound can "nearly" be achieved, as formalized by the following result.

Theorem 2. *Under the same assumptions used for the independent valuations result in Theorem 1, and with the additional assumption that* $\mathcal{R}_i = \mathcal{A}_i \times \mathcal{A}_i$ *for* $i \in \{1, 2, \ldots, N\}$*, for any strictly increasing functions* $\Gamma_i(\cdot)$ *defined on* $\mathcal{A}_i^d$ *such that* $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$ *for all* $i \in \{1, 2, \ldots, N\}$*, the distribution of valuations* $(\Gamma_1(A_1), \Gamma_2(A_2), \ldots, \Gamma_N(A_N))$ *is such that there is a corresponding Bayesian Nash equilibrium using weakly increasing strategies with the same distribution of actions as* $P(A)$*. Moreover, for any* $\epsilon > 0$*, there are such* $\Gamma_i(\cdot)$ *with the further property that* $0 \leq \sup_{a_i \in \mathcal{A}_i} (\Gamma_i(a_i) - \Gamma_{Li}(a_i)) < \epsilon$ *and there are such* $\Gamma_i(\cdot)$ *with the further property that* $0 \leq \sup_{a_i \in \mathcal{A}_i} (\Gamma_{Ui}(a_i) - \Gamma_i(a_i)) < \epsilon$.

The condition $\mathcal{R}_i = \mathcal{A}_i \times \mathcal{A}_i$ requires every pair of actions has game-structure identification of differences. This is needed to ensure that the econometrician knows the difference in expected allocation and expected transfer for any possible pair of actions, to ensure it is possible to establish optimality of a given action for a given valuation, as required in a sharpness proof. Sufficient conditions would be, essentially, that the econometrician *ex ante* knows the expected allocations and expected transfers (or the allocation rule and the transfer rule), or that all actions are used in the data.

The distribution of valuations $(\Upsilon_{L1}(A_1), \Upsilon_{L2}(A_2), \ldots, \Upsilon_{LN}(A_N))$ or the distribution of valuations $(\Upsilon_{U1}(A_1), \Upsilon_{U2}(A_2), \ldots, \Upsilon_{UN}(A_N))$ that comprise exactly the lower bound and upper bound from Theorem 1 may not be achievable in a Bayesian Nash equilibrium because they might require that two or more different actions are used by a single valuation, if the lower (upper) bound is the same for at least two different actions, which cannot happen in equilibrium using pure strategies as given in Assumption 7 (Weakly increasing strategy is used). However, per the last part of Theorem 2, the lower and upper bounds can be approached arbitrarily closely, and thus are essentially "limit points" of the identified set.

Under Assumption 1 (Dependent valuations), the identification result in Theorem 1 appears to not share this sharpness property, and it appears quite difficult to provide a useful[6] characterization of the sharp identified set with dependent valuations, as a consequence of the need to bound player beliefs. Still, there is a sense in which the identification result is "sharp in the limit" in that it limits to point identification in particular when the action space either is or limits to a continuous/interval action space, per Section 3.7 and Appendix A.

3.6. **Sketch of identification proof.** Under Assumption 4 (Optimal strategy is used), for any valuation θ_i, any action $\tilde{a}_i(\theta_i)$ that solves the utility maximization problem in Equation 1 satisfies

$$\theta_i E_{\Pi_i}(\overline{x}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)\right). \tag{8}$$

Under Assumption 5 (Correct beliefs), Equation 8 implies

$$\theta_i E_P(\overline{x}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)\right). \tag{9}$$

Under Assumption 7 (Weakly increasing strategy is used), for any action $a_i^* \in \mathcal{A}_i$ there is an interval

$$\Theta_i(a_i^*) = \{\theta_i : a_i(\theta_i) = a_i^*\} \tag{10}$$

of valuations such that player i with valuation θ_i uses action a_i^* if and only if $\theta_i \in \Theta_i(a_i^*)$. Moreover, if $a_i \neq a_i'$ then $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are disjoint; and if $a_i < a_i'$ and $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are both non-empty then $\sup \Theta_i(a_i) \leq \inf \Theta_i(a_i')$. Therefore, for any $z_i, z_i' \in \mathcal{A}_i^d$,

$$E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{x}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{x}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')) \tag{11}$$

$$E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{t}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')). \tag{12}$$

[6]Of course, it is always possible to trivially write down the identified set by its definition that it is a distribution of valuations consistent with the data and the assumptions.

Hence, the beliefs expressions in Equation 9 conditioning on θ_i are generically not point identifiable, because generically multiple valuations use any given $z_i' \in \mathcal{A}_i$.

Equation 9 implies, under Assumptions 5 (Correct beliefs), 6 (Counterfactual *ex interim* expected utility maximization problem has a solution), 7 (Weakly increasing strategy is used), and 8 (Monotone effect of counterfactual beliefs on utility), for $\theta_i' < \theta_i < \theta_i''$, and letting $a_i(\theta_i; \theta_i'')$ be the selection per Assumption 8, for any $z_i \in \mathcal{A}_i$,

$$\begin{aligned}(13)\qquad & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i') \geq \\ & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) \geq \\ & \qquad \theta_i E_P(\overline{x}_i(a_i(\theta_i;\theta_i''), A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i(\theta_i;\theta_i''), A_{-i})|\theta_i) \geq \\ & \qquad \theta_i E_P(\overline{x}_i(a_i(\theta_i;\theta_i''), A_{-i})|\theta_i'') - E_P(\overline{t}_i(a_i(\theta_i;\theta_i''), A_{-i})|\theta_i'') \geq \\ & \qquad \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'') - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'').\end{aligned}$$

Then, for any $z_i \in \mathcal{A}_i$, and letting $z_i' < a_i(\theta_i) < z_i''$ be any two actions that are actually used by player i, for at least some valuation of player i, i.e., $z_i', z_i'' \in \mathcal{A}_i^d$:

$$\begin{aligned}(14)\qquad & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') \\ & = \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) \\ & \geq \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_{P_i}(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) \\ & \geq \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) \\ & = \theta_i E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'').\end{aligned}$$

And consequently,

$$\begin{aligned}(15)\qquad & \theta_i \geq \frac{E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\ & \qquad \forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d, \\ & \qquad\quad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\} \\ & \theta_i \leq \frac{E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\ & \qquad \forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d,\end{aligned}$$

$$z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}$$

An implication of Equation 15, restricted to specifications with game-structure identification of differences, is

$$\begin{aligned}
(16)\qquad & \theta_i \geq \frac{E_P(\bar{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\
& \forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\} \\
& \quad (a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i \\
& \theta_i \leq \frac{E_P(\bar{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\
& \forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\} \\
& \quad (a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i
\end{aligned}$$

Consequently, the valuation corresponding to a_i must be between $\Phi_{Li}(a_i)$ and $\Phi_{Ui}(a_i)$. By another application of Assumption 7 (Weakly increasing strategy is used), any valuation consistent with a_i is between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$.

3.7. **Results with increasing number of actions.** Consider the limit of $z_i \to a_i, z_i' \uparrow a_i, z_i'' \downarrow a_i$ in the right hand sides of Equations 2-3. This limit can arise when the action space is such that the action a_i is in the interior of the action space. Also, this limit can approximate a (heuristic) limit when the number of actions increases to the limit of a continuous/interval action space, with the caveat that the game itself changes when the action space changes, so such a limit cannot be taken literally without a careful analysis of how the game changes. A formal point identification result with an interval action space is provided in Appendix A.

A sketch of the intuition for how point identification arises in the limit goes as follows. Note that

$$\begin{aligned}
&\frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \xrightarrow{z_i' \uparrow a_i \text{ and } z_i'' \downarrow a_i} \\
&\frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = a_i) - E_P(\bar{t}_i(z_i, A_{-i})|A_i = a_i)}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i) - E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i)} = \\
&\frac{\frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = a_i) - E_P(\bar{t}_i(z_i, A_{-i})|A_i = a_i)}{a_i - z_i}}{\frac{E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i) - E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i)}{a_i - z_i}} \xrightarrow{z_i \to a_i} \frac{\left.\frac{\partial E_P(\bar{t}_i(z_i, A_{-i})|A_i = a_i)}{\partial z_i}\right|_{z_i = a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i)}{\partial z_i}\right|_{z_i = a_i}}.
\end{aligned}$$

The first limit requires continuity of the conditional expectations as a function of the conditioning variable, so that $E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') \to E_P(\bar{t}_i(a_i, A_{-i})|A_i = a_i)$ and $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') \to E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i)$ as $z_i' \uparrow a_i$ and $E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'') \to E_P(\bar{t}_i(z_i, A_{-i})|A_i = a_i)$ and $E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') \to E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i)$ as $z_i'' \downarrow a_i$, where the third and fourth limits must hold uniformly over z_i since z_i is part of the limiting sequence.[7] The second limit is an application of the definition of the derivative, and requires that the derivatives exist and that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i} \neq 0$. In that case, the valuation θ_i corresponding to action a_i is bounded above and below by, and thus must equal, $\frac{\left.\frac{\partial E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}} = \Psi_i(a_i)$.[8] In particular, this suggests that relatively finer discrete action spaces (e.g., auctions that allow bids that are any integer multiple of one cent compared to any integer multiple of five dollars) can be expected to result in relatively tighter identification of the distribution of valuations.

4. Conclusions

This paper develops identification results for the distribution of valuations in a class of allocation-transfer games that determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The identification results are constructive and are based on the assumption of monotone equilibrium. The results allow dependent valuations,

[7]Continuity of the conditional expectations is related to the condition of no point masses used in Appendix A. Suppose $a_i(\theta_i) = a_i^*$ has the unique solution θ_i^*, so θ_i^* is the unique valuation to use action a_i^*. Then there will be no point mass at a_i^* in the distribution of A_i. Suppose further that $a_i(\cdot)$ is *strictly* increasing in a neighborhood of θ_i^*, and that $a_i(\cdot)$ is continuous in a neighborhood of θ_i^*. The first condition is slightly stronger than the condition that θ_i^* is the unique valuation to use action a_i^*, since it could otherwise be that, for example, $a_i(\cdot)$ is strictly increasing "below" θ_i^*, has a jump discontinuity at θ_i^*, and is flat "above" θ_i^*. Since $a_i(\cdot)$ is weakly increasing per Assumption 7, $a_i(\cdot)$ is continuous except for a countable set. Then, for example, $E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') = E_P(\bar{t}_i(a_i, A_{-i})|\theta_i = a_i^{-1}(z_i'))$. Supposing that $E_P(\bar{t}_i(a_i, A_{-i})|\theta_i) = E_{\Pi_i}(\bar{t}_i(a_i, a_{-i})|\theta_i)$ is itself continuous as a function of θ_i, it would follow that $E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') \to E_P(\bar{t}_i(a_i, A_{-i})|A_i = a_i)$ as $z_i' \to a_i$ and similarly for the other limits of the other conditional expectations. Otherwise, if there were multiple valuations to use action a_i, resulting in a point mass at a_i, a "small change" in conditioning on $A_i = a_i$ versus $A_i = z_i'$ could result in a "large change" in the actual expected value, since it would correspond to a "large change" in the set of θ_i being equivalently conditioned on.

[8]This heuristic analysis also implicitly assumes game-structure identification on the right hand side of Equations 2-3. Further, under the condition that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i} \neq 0$, assume that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i}$ is continuous in z_i (i.e., continuously differentiable). Consider the case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i} > 0$ on an interval neighborhood of a_i. The case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i} < 0$ on an interval neighborhood of a_i would be similar, though it seems inconsistent with Assumption 7. Then $E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i)$ would be strictly increasing at $z_i = a_i$, and hence (when $z_i' \approx a_i \approx z_i''$), $z_i < a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0$ in the right hand side of Equation 2 and $z_i > a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0$ in the right hand side of Equation 3.

discrete parts of the action space, and non-differentiability, and unknown (to the econometrician) details of how the allocations and transfers are determined.

Appendix A. Point identification in the limit

As noted in Section 3.7, the partial identification result "limits" to point identification under certain conditions. This section formalizes that result.

Assumption 10 (Continuous action space and no point masses in distribution of actions)**.** *For each $i \in \{1, 2, \ldots, N\}$, $\mathcal{A}_i = [\alpha_i, \beta_i]$ and there are no point masses in the observed distribution of actions of player i.*

Compared to Assumption 3 (Action space is ordered), Assumption 10 rules out discrete actions.

Assumption 11 (Smooth distribution of valuations)**.** *The distribution $F(\cdot)$ has associated ordinary density $f(\cdot)$. For each $i \in \{1, 2, \ldots, N\}$, the support of the distribution of θ_i is an interval.*

Under Assumptions 1 (Dependent valuations), 7 (Weakly increasing strategy is used), and 11 (Smooth distribution of valuations), the lack of point masses from Assumption 10 (Continuous action space and no point masses in distribution of actions) is equivalent to the condition that the strategy is *strictly* increasing.[9]

Assumption 12 (Differentiable *ex interim* expected allocation and transfer)**.** *For each $i \in \{1, 2, \ldots, N\}$, there is a set $\mathcal{E}_{i,d}$ with $P(A_i \in \mathcal{E}_{i,d}) = 0$ such that for each possible valuation θ_i, the expected allocation $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)$ and the expected transfer $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)$ are differentiable functions of a_i, evaluated at any $a_i^* \in$ support$(a_i(\theta_i)) \cap \mathcal{E}_{i,d}^C$.*

The notation S^C for some set S is the complement of the set S. Assumption 12 requires that *ex interim* expected allocation and *ex interim* expected transfer given valuation θ_i are differentiable on the support of the strategy $a_i(\theta_i)$. Intuitively, this corresponds to the existence of the derivatives used in the heuristic argument in Section 3.7. Under Assumption 7 (Weakly increasing strategy is

[9] If two valuations use the same action, then there is a point mass at that action because the entire interval connecting those valuations would also use that same action. So, if there are no point masses, then no two valuations use the same action, so the strategy must indeed be *strictly* increasing. Conversely, obviously if the strategy is *strictly* increasing, then there are no point masses in the distribution of actions by Assumptions 1 and 11. This conclusion is not true without Assumption 7, since if $a_i(\cdot)$ were non-monotone, then the set $\{\theta_i : a_i(\theta_i) = a_i^*\}$ can be non-singleton, but not necessarily of positive probability under the distribution of θ_i. Therefore, if the strategy were non-monotone, then multiple valuations could use the same action a_i^* even though there is no mass point at a_i^*.

used), $a_i(\theta_i)$ is a degenerate random variable (i.e., a pure strategy). However, under Assumption 4 (Optimal strategy is used) alone, mixed strategies are allowed. As mentioned above, breaking up the assumptions in this way makes it easier to refer to the separate roles of the assumptions. Assumption 12 allows a probability zero exceptional set of actions at which differentiability fails.

Let

$$\Psi_i^x(z) \equiv \left.\frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \text{ and } \Psi_i^t(z) \equiv \left.\frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \tag{17}$$

and let

$$\Psi_i(z) \equiv \frac{\Psi_i^t(z)}{\Psi_i^x(z)}. \tag{18}$$

The proof of Theorem 3 shows that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ actually do exist for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,d}^C$.

Definition 5 (Action with game-structure identification of derivatives)**.** An action a_i is an action with game-structure identification of derivatives if $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ can be identified to exist, and $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified quantities. Per convention, identification of derivatives on the boundary of $\mathcal{A}_i$ is understood to concern identification of the corresponding *one-sided* derivative.

Assumption 13 (Game-structure identification of derivatives)**.** *For each $i \in \{1, 2, \ldots, N\}$, there is a set $\mathcal{E}_{i,r}$ with $P(A_i \in \mathcal{E}_{i,r}) = 0$ such that if $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,r}^C$ is such that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist then a_i is an action with game-structure identification of derivatives per Definition 5.*

Assumption 13 requires game-structure identification of derivatives for all actions used in the data except for the probability zero exceptional set $\mathcal{E}_{i,r}$. This accommodates the possibility that game-structure identification of derivatives may fail on a set of probability zero. Similar to game-structure identification of differences from Definition 4, game-structure identification of derivatives follows from standard conditions on identification/estimation of derivatives of conditional expectations. See Appendix E for details.

Assumption 14 (Non-zero marginal expected allocation)**.** *For each $i \in \{1, 2, \ldots, N\}$, there is a set $\mathcal{E}_{i,m}$ with $P(A_i \in \mathcal{E}_{i,m}) = 0$ such that $\Psi_i^x(a_i) \neq 0$ for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,m}^C$.*

Assumption 14 allows a probability zero exceptional set $\mathcal{E}_{i,m}$.

Theorem 3. *Under Assumptions 1 (Dependent valuations), 3 (Action space is ordered), 4 (Optimal strategy is used), 5 (Correct beliefs), 7 (Weakly increasing strategy is used), 10 (Continuous action space and no point masses in distribution of actions), 11 (Smooth distribution of valuations), 12 (Differentiable* ex interim *expected allocation and transfer), 13 (Game-structure identification of derivatives), and 14 (Non-zero marginal expected allocation), the distribution of valuations θ is point identified, and the identification is constructive, because the distribution of θ equals the distribution of $(\Psi_1(A_1), \Psi_2(A_2), \ldots, \Psi_N(A_N))$, where $(A_1, A_2, \ldots, A_N)$ is distributed according to the data $P(A, X, T)$ and $\Psi_i(\cdot)$ is the identifiable function given in Equation 18.*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations) and replace the Ψ functions with the Λ functions defined in Equation 20.*

Let

$$\Lambda_i^x(z) \equiv \left.\frac{\partial E_P(\overline{x}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z} \text{ and } \Lambda_i^t(z) \equiv \left.\frac{\partial E_P(\bar{t}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z}. \tag{19}$$

Also, let

$$\Lambda_i(z) \equiv \frac{\Lambda_i^t(z)}{\Lambda_i^x(z)}. \tag{20}$$

Then,

$$\Lambda_i^x(z) = \left.\frac{\partial E_P(X_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z} \text{ and } \Lambda_i^t(z) = \left.\frac{\partial E_P(T_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z}. \tag{21}$$

Under Assumption 2 (Independent valuations), the econometrician can identify $\Lambda_i^x(\cdot)$ and $\Lambda_i^t(\cdot)$ using the expressions in Equation 21. ★

Appendix B. Examples of games

The class of allocation-transfer games studied in this paper is illustrated via examples.

Example 1 (Contests)**.** In contest models, the actions are interpreted as "costly effort" toward winning a valuable object. The economic theory of such models has been developed in, for example, Hillman and Riley (1989), Baye, Kovenock, and De Vries (1993), Amann and Leininger (1996), Krishna and Morgan (1997), Lizzeri and Persico (2000), and Parreiras and Rubinchik (2010), in addition to an overall large literature. See for example Konrad (2007, 2009) for a summary of the literature, including discussion of theoretical applications to a broad range of instances of competition, including advertising, litigation, political lobbying, research and development, and sports. Wasser (2013) applies Athey (2001) to establish conditions for a monotone equilibrium in contests.

The valuation θ_i is the value that player i has for the object. Often, the "efforts" are equivalent to financial expenditures, so that $\mathcal{A}_i = [0, \infty)$ and the transfer rule is $\bar{t}_i(a) = a_i$. However, other transfer rules are also possible. For example, it might be that part of the effort is "refundable," so that players only expend some fraction of their effort, possibly depending on whether the player wins or loses (e.g., see the models in Riley and Samuelson (1981) and Matros and Armanios (2009)). The allocation rule $\overline{x}(a) = (\overline{x}_1(a), \overline{x}_2(a), \ldots, \overline{x}_N(a))$ is known as the "*contest success function*" that relates the actions taken by the players to the probabilities that each of the players wins the valuable object. The econometrician may not know the contest success functions $\overline{x}(\cdot)$, and indeed the economic theory literature has explored a variety of different contest success functions. See for example Corchón and Dahm (2010) for a detailed discussion. For example, following Tullock (1980)-style models, $\overline{x}_i(a) = \begin{cases} \frac{a_i^r}{\sum_{j=1}^N a_j^r} & \text{if } a \neq 0 \\ \frac{1}{N} & \text{if } a = 0 \end{cases}$ for some $r > 0$. In particular, the case of $r = 1$ has been interpreted as a "lottery" in which the probability that player i wins is equal to player i's share of the overall aggregate effort. The specification states that if all players expend no effort, then each player has equal chance of winning the contest. More generally, there can be functions $f_i(\cdot)$ such that $\overline{x}_i(a) = \frac{f_i(a_i)}{\sum_{j=1}^N f_j(a_j)}$, including the logistic specification $f_i(z) = e^{kz}$ as in Hirshleifer (1989). Alternatively, following Lazear and Rosen (1981)- and Dixit (1987)-style models, $\overline{x}_i(a) = P_\epsilon(a_i + \epsilon_i > \max_{j \neq i}(a_j + \epsilon_j))$, where P_ϵ is the distribution of "noise" or "randomness" involved in determining the contest winner. Because the identification results do not require a complete specification of the game, the identification results do not require the econometrician to know $\overline{x}(\cdot)$ (or the underlying distribution $\widetilde{x}(\cdot)$). In particular, the econometrician might not know r or f_i or P_ϵ.

In the above specifications, generally a player that expends the most effort is most likely to win, but is not guaranteed to win. In the limiting case of the "all-pay auction" formulation,

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } i \text{ expends the most effort} \\ p_i(a) & \text{if } i \text{ ties for expending the most effort with at least one other player} \\ 0 & \text{if } i \text{ does not expend the most effort,} \end{cases}$$

where $p_i(a)$ reflects the tie-breaking rule. In all-pay auction models, the player that expends the most effort is guaranteed to win.

Example 2 (Auctions)**.** Auction models can involve various complications like "participation costs," reserve prices, asymmetries, and/or multiple units possibly with endogenous supply. The economic theory of auctions has been reviewed, for example, in Klemperer (1999, 2004), Milgrom (2004), Menezes and Monteiro (2005), and Krishna (2009). Specifically the economic theory of auctions with a discrete action space has been developed in Chwe (1989), Rothkopf and Harstad (1994), Dekel and Wolinsky (2003), David, Rogers, Jennings, Schiff, Kraus, and Rothkopf (2007). One feature of the auction theory literature is the range of auction formats, implying a range of allocation and transfer rules. Much of the economic theory literature has focused on establishing monotonicity of the strategy in auction models, and moreover the literature on general conditions for monotone equilibrium in games often treats auctions as a leading example of their results.

The valuation θ_i is the value player i has for a unit of the object being auctioned. The specific auction format would be reflected in the allocation rule $\overline{x}(\cdot)$ and transfer rule $\overline{t}(\cdot)$, and the identification strategy can apply to a wide range of auction formats. The general allocation-transfer game framework flexibly accommodates various auction formats.

Let $H_i(a) = \max_{j \neq i \text{ and } j \text{ s.t. } a_j \geq r_j} a_j$ be the highest bid other than the bid of player i, among the bids from players that exceed the corresponding reserve price, where $r_i \geq 0$ is the reserve price for player i.

Also let $S(a)$ be the quantity allocated to the winning bidder as a function of the profile of bids (e.g., Milgrom (2004, Section 4.3.3)). For example, the supply $S(a)$ might depend only on the winning bid, as in a "supply curve" at the "price" of the winning bid. See also Example 3 for related models where $S(a)$ can be interpreted as a "demand curve." The standard case that there is one exogenous unit of the object being auctioned is the special case that $S(\cdot) \equiv 1$.

The allocation is the awarding of units of the object from the auction. Then, for example, in auction formats where the highest bidder wins, as long it exceeds its reserve price and the highest competitor's bid among those bids exceeding the corresponding reserve price,

$$\overline{x}_i(a) = \begin{cases} S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i, \end{cases}$$

where $p_i(a) \in [0, S(a)]$ reflects the tie-breaking rule, the expected number of units that player i is allocated when bids are a, involving a tie for high bid.

The transfers include the payments to the auctioneer, but could include other transfers, like participation costs[10] when applicable. The transfer rule also depends on the auction format. For example in a first price auction, and noting that $\bar{t}_i(a)$ is the *expected* transfer that integrates over the tie-breaking rule,

$$\bar{t}_i(a) = \begin{cases} a_i S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ a_i p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i \end{cases}$$

Other auction formats would have different allocation rules and/or transfer rules.

The econometrician may not know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$, because the econometrician may not know the "supply function" $S(a)$, but again, the identification results do not require the econometrician to know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$.

Because the allocation-transfer game framework does not necessarily require the assumption of symmetric players, the auction could involve such asymmetries as "strong" and "weak" bidders, as in Milgrom (2004, Section 4.5). For example, Campo, Perrigne, and Vuong (2003) have focused on establishing point identifying assumptions for asymmetric bidders with affiliated private values in first price auctions. Reny and Zamir (2004) have studied the existence of monotone equilibrium in related auction models.

[10]A participation cost can be modeled in a few different ways, particularly concerning whether or not the players know their own valuation at the time they make the participation decision. A third approach allows that bidders observe a signal of their valuation at the time of their participation decision, an identification problem studied in Gentry and Li (2014). Other identification results emphasizing entry/participation in particular auction models includes Marmer, Shneyerov, and Xu (2013) (focusing on identifying the selection effect, and discriminating between models of entry), Fang and Tang (2014) (focusing on inferring bidder risk attitudes), and Li, Lu, and Zhao (2015) (focusing on testable implications of risk aversion). The economic theory of auctions with participation costs has been developed in, for example, Samuelson (1985), McAfee and McMillan (1987), Levin and Smith (1994), Tan and Yilankaya (2006), and Cao and Tian (2010). See for example (Krishna, 2009, Section 2.5) for equilibrium in auctions with reserve prices.

Henderson, List, Millimet, Parmeter, and Price (2012) and Luo and Wan (2018) explore the impact of monotonicity of the bidding strategy in specific first-price auction models with independent valuations on the properties of the estimator (e.g., rate of convergence, optimality, etc.), whereas this paper explores the role of monotonicity in identification.

Haile and Tamer (2003) study the (partial) identification of bidder valuations that arises when the econometrician has an incomplete model, specifically in an incomplete model of English auctions with symmetric independent private values. See also Chesher and Rosen (2017) for further identification results in a related model, based on generalized instrumental variables. Haile and Tamer (2003) studied identification of bidder valuations based on the assumptions that bidders will not be "outbid" and will not "overbid."

Another important identification problem, also leading to partial identification, particularly in certain auction formats, concerns the "missing data" problem when the econometrician does not observe the bids of all of the players. Aradillas-López, Gandhi, and Quint (2013) have established partial identification in the important case of an ascending auction with correlated valuations, focusing on showing partial identification of economically relevant seller profit and bidder surplus quantities rather than the object in this paper, the overall joint distribution of valuations. Because the data used by the identification strategy developed here includes the actions of all players, it cannot be applied to address the identification problem studied in Aradillas-López, Gandhi, and Quint (2013). However, the identification strategy developed here does allow "missing data" on other parts of the game, for example the "participation cost" in an auction with a participation cost. Similarly, because the identification strategy can apply to an incomplete specification of the model, the identification results also accommodate "missing *ex ante* knowledge," for example on endogenous quantity functions in an auction. Tang (2011) focuses on partial identification of auction revenue in first-price auctions with common values, which also is not addressed by this paper, which assumes private values.

Example 3 (Procurement auctions, reverse auctions, oligopoly models, etc.)**.** Models of procurement auctions, reverse auctions, and related situations are similar to auctions, with the distinguishing feature that the N players are bidding to *sell* units of an object, rather than *buy* units of an object. Therefore, the valuation θ_i can be interpreted to be player i's (constant) marginal cost of supplying one unit of the object, and the "low bid" wins the market. Let $L_i(a) = \min_{j\neq i \text{ and } j \text{ s.t. } a_j \leq r_j} a_j$ be the lowest bid other than the bid of player i, among the bids from players that are below the corresponding reserve price. The "allocation" experienced by player i is the quantity of the object that player i supplies, and therefore the allocation is negative, so the allocation rule could be

$$\overline{x}_i(a) = \begin{cases} -S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i, \end{cases},$$

where, similarly to Example 2, $S(a)$ is the endogenous quantity (i.e., "demand") given the profile of bids a, r_i is the maximum acceptable bid for player i, and $p_i(a)$ reflects the tie-breaking rule. The "transfer" experienced by player i is the payment to player i. Due to the convention in this paper that transfers are *from* the player, transfers are negative. For example, it could be that

$$\overline{t}_i(a) = \begin{cases} -a_i S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -a_i p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i \end{cases}$$

Some models of oligopoly competition are basically the same game, with N firms in an oligopoly having privately known constant marginal costs of production competing to win the oligopoly market, see for example Vives (2001, Chapter 8). In these models, the "endogenous quantity" $S(a)$ is the demand curve, generally depending on the lowest bid (i.e., the "realized price"). As with the endogenous supply in Example 2, the econometrician may not know the "demand curve" and therefore again not know $\overline{x}_i(a)$ and/or $\overline{t}_i(a)$, but again the identification results do not require the econometrician to know $\overline{x}_i(a)$ and/or $\overline{t}_i(a)$.

Example 4 (Partnership dissolution)**.** The economic theory of partnership dissolution has been developed in Cramton, Gibbons, and Klemperer (1987), in addition to a huge subsequent literature. There are N co-owners of an object. Prior to partnership dissolution, player i owns share r_i of the object and has valuation θ_i for the object. The econometrician need not know these ownership shares.

In the "bidding game" formulation of partnership dissolution developed in Cramton, Gibbons, and Klemperer (1987), there are initial transfers between the co-owners that depend on their ownership shares. Since these initial transfers do not depend on valuations, they are not revealing of valuations. In the special case of equal ownership shares, these initial transfers are zero. In any case, the econometrician need not observe data on these initial transfers in order to apply the identification strategy. Indeed, the identification strategy does not rely on the game implementing such initial transfers. These initial transfers are for purposes of satisfying the individual rationality constraint, violation of which does not change the identification strategy in this paper, since this paper essentially only uses the incentive compatibility constraint. See formula C of Cramton, Gibbons, and Klemperer (1987, Theorem 2). Then, each co-owner bids for ownership, so the action in the game are bids, with the highest bidder winning ownership. The transfer from player i is (omitting the "lump sum" initial transfer reflecting ownership shares but not valuations): $\bar{t}_i(a) = a_i - \frac{1}{N-1}\sum_{j\neq i}^{N} a_j$, so player i transfers its bid even if it loses, and is "compensated" by the bids of the other players.

Example 5 (Public good provision)**.** In models of the provision of public goods or public projects, the distinguishing feature is that the allocation is the same to all players, reflecting the "public" nature of the object. The valuation θ_i reflects the private value that player i places on the public good. The economic theory of such models has been developed in Bergstrom, Blume, and Varian (1986), Bagnoli and Lipman (1989), Mailath and Postlewaite (1990), Alboth, Lerner, and Shalev (2001), Menezes, Monteiro, and Temimi (2001), and Laussel and Palfrey (2003), in addition to a huge overall literature, summarized for example in Ledyard (2006). See Lemma 1 or the discussion of "regular" equilibrium in Laussel and Palfrey (2003) for the role of monotonicity in the strategies. Or see the characterization of the equilibrium strategies in Menezes, Monteiro, and Temimi (2001). In direct revelation games (e.g., Clarke (1971)-Groves (1973) games), players report their valuation, in which case the identification problem is trivial. However, in other games, the actions of the players are interpreted as contributions to the public good, and the object is allocated (e.g., the public project is completed) if and only if the sum of the contributions of the players is greater than the cost of producing the public good. The contributions may or may not be refunded if the public good is not produced, depending on the specific game. See for example Menezes, Monteiro, and Temimi (2001). Some models of public good provision, along the lines of Palfrey and Rosenthal (1984) (who worked with complete information), involve a discrete and even binary action space (contribute an *ex ante* fixed amount or not).

Example 6 (Strategic (non-"price taking") market behavior)**.** Models of strategic (non-"price taking") market behavior, specifically models based on multilateral double auctions, involve N_s sellers (i.e., players that currently each own a unit of the object) and N_b buyers (i.e., players that potentially would each like to buy a unit of the object). The buyers and sellers interact in order to trade units of the object in exchange for monetary payments. The economic theory of such models has been developed in Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), and Wilson (1985), in addition to a huge subsequent literature. See Fudenberg, Mobius, and Szeidl (2007), Kadan (2007), or Araujo and De Castro (2009) for recent results. See Bolton and Dewatripont (2005, Chapter 7) for a textbook treatment. For monotonicity in the equilibrium strategies, see e.g., Chatterjee and Samuelson (1983, Theorem 1) and Satterthwaite and Williams (1989a, Definition of "regular" equilibrium) and Fudenberg, Mobius, and Szeidl (2007, Theorem 1). The case of $N_s = 1 = N_b$ has seen particular attention, as models of bilateral trade.[11] The case of $N_s > 1$ and $N_b > 1$ has also seen particular attention, as "strategic" versions of supply and demand models, in which individual market participants do not act as competitive price takers. Although the theory literature has tended to treat these two cases separately, the identification strategy can accommodate both cases.

The valuation of player i for a unit of the object is the private information θ_i. The buyers announce "bid prices" and the sellers announce "ask prices" and trade proceeds. Suppose that $a_{(N_s)}$ is the N_s-th highest announcement and $a_{(N_s+1)}$ is the $N_s + 1$-st highest announcement, both among the combined set of announcements (i.e., bids and asks) from buyers and sellers. Let $z(a) = ka_{(N_s)} + (1-k)a_{(N_s+1)}$ be the resulting transaction price, where $k \in [0,1]$ is a parameter of the model that might either be known or unknown by the econometrician (an example of a possibly incomplete specification of the model of the game). Then one possible allocation rule and transfer rule is

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } a_i > z(a) \\ p_i(a) & \text{if } a_i = z(a) \\ 0 & \text{if } a_i < z(a) \end{cases} \text{ and } \overline{t}_i(a) = \begin{cases} z(a) & \text{if i is a buyer and } a_i > z(a) \\ -z(a) & \text{if i is a seller and } a_i < z(a) \\ p_i(a)z(a) & \text{if i is a buyer and } a_i = z(a) \\ -(1-p_i(a))z(a) & \text{if i is a seller and } a_i = z(a) \\ 0 & \text{otherwise,} \end{cases}$$

where $p_i(a)$ reflects a tie-breaking rule with the condition that $\sum_{i=1}^{N} \overline{x}_i(a) = N_s$ for all a. In particular, in the generic case of $a_{(N_s)} > a_{(N_s+1)}$, the tie-breaking rule is such that $p_i(a) = 1$ when $a_i = z(a)$ and

[11]There are a variety of different "bilateral trade" or "bargaining" models, not all of which proceed in the same way. For example, Merlo and Tang (2012) study identification of a different bargaining model that evidently does not fit this allocation-transfer game framework.

$k = 1$ and $p_i(a) = 0$ when $a_i = z(a)$ and $k = 0$. Therefore, ignoring ties by considering the generic situation that $a_{(N_s)} > a_{(N_s+1)}$, and because $a_{(N_s)} \geq z(a) \geq a_{(N_s+1)}$ with at least one inequality strict, the players with the N_s highest announcements, among both buyers and sellers, are allocated a unit of the object. The transaction price is $z(a)$, and buyers that are allocated a unit of the object pay $z(a)$ and sellers that are not allocated a unit of the object receive $z(a)$. See for example Fudenberg, Mobius, and Szeidl (2007) for more details. These allocation and transfer rules might be unknown by the econometrician, if the econometrician does not know k, in which case the identification strategy involves identifying the allocation and transfer rules directly from the data.

The main assumption of the identification strategy is that the players use monotone strategies. For buyers, this requires that buyers announce that they are willing to pay relatively more for a unit of the object when their valuation for a unit of the object is relatively higher. For sellers, this requires that sellers announce that they require a relatively higher payment for a unit of the object when their valuation for a unit of the object is relatively higher. Further, equilibrium strategies can be difficult to characterize (e.g., Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989a)), making it useful that assuming a property of the equilibrium is sufficient for the identification strategy, without needing to explicitly characterize the equilibrium solution. For example, in one particular case (with $k = 0$ and other assumptions), Satterthwaite and Williams (1989b) show that the equilibrium strategy for the buyers is the solution to a differential equation involving a combinatorial expression involving the unknown distribution of valuations. Chatterjee and Samuelson (1983, Example 2) show in a specific example with $N_s = 1 = N_b$ that the strategy for the buyer or seller can involve a "flat spot" if the support of the distribution of valuations for the buyer is different from the support of the distribution of valuations for the seller, even with a continuous action space. Leininger, Linhart, and Radner (1989) show that there exists equilibria in which both buyers and sellers use step functions as their strategies. One of these equilibria is particularly simple, with the valuations supported on $[0, 1]$. For some $\overline{\theta}$, a buyer with a valuation less than $\overline{\theta}$ bids 0 and a buyer with a valuation weakly greater than $\overline{\theta}$ bids $\overline{\theta}$. Conversely, a seller with a valuation weakly less than $\overline{\theta}$ asks $\overline{\theta}$ and a seller with a valuation greater than $\overline{\theta}$ asks 1. The corresponding *ex interim* expected allocation and *ex interim* expected transfer would not be differentiable.

Appendix C. Proofs

In order to economize on space, references to equations and other quantities already defined in the body of the paper are used in the proofs.

Proof of Lemma 1. By Assumption 5 (Correct beliefs), $\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i') = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i')$, because the distribution of $A_{-i}|\theta_i'$ is the same as the distribution of $a_{-i}(\theta_{-i})|\theta_i'$. Under Assumption 7 (Weakly increasing strategy is used), and the condition that $\theta_i\overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i}) - \overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})$ is a weakly decreasing function of a_{-i}, $\theta_i\overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i}(\theta_{-i})) - \overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i}(\theta_{-i}))$ is a weakly decreasing function of θ_{-i}. Under affiliation, by standard properties of affiliated random variables (e.g., Milgrom (2004, Theorem 5.4.5)), it follows that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i')$ is a weakly decreasing function of θ_i'. Alternatively, under monotonicity of $\theta_{-i}|\theta_i$ in the usual multivariate stochastic order, by standard properties of the usual multivariate stochastic order (e.g., Shaked and Shanthikumar (2007, Chapter 6)), it follows that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i') \geq \theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i;\theta_i''), a_{-i})|\theta_i'')$ for $\theta_i' \leq \theta_i''$. □

Proof of Lemma 2. By definition, $\overline{x}_i(a) = E(\tilde{x}_i(a)) = E(\tilde{x}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a) = E(\tilde{t}_i(a)) = E(\tilde{t}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$. Under the conditions of the lemma, for $a = (a_1, a_2, \ldots, a_N)$ such that $a_j \in \mathcal{A}_j^d$ for all j, it holds that also $a \in \mathcal{A}^d$ and therefore $\overline{x}_i(a)$ and $\overline{t}_i(a)$ are point identified by the previous expressions in terms of conditional expectations, conditional on a discrete variable. Then, consider $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$ and suppose that $a_i \in \mathcal{A}_i^d$ and $z_i' \in \mathcal{A}_i^d$. Obviously, the support of $A_{-i}|(A_i = z_i')$ is a subset of the support of A_{-i}, and $a_i \in \mathcal{A}_i^d$ by assumption, and therefore $\overline{x}_i(a_i, a_{-i})$ is point identified at all points relevant to $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$. And of course the distribution of $A_{-i}|(A_i = z_i')$ is identified since $z_i' \in \mathcal{A}_i^d$. Therefore, $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$ is point identified. It is similar for $E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$, $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i')$, and $E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$. Therefore, there is game-structure identification of differences per Definition 4. □

Proof of Theorem 1. By Assumption 4 (Optimal strategy is used), Equation 8 is a necessary condition for any action $\tilde{a}_i(\theta_i)$ used by player i. Then, under Assumption 5 (Correct beliefs), Equation 9 is an equivalent necessary condition. Then, under Assumptions 6 (Counterfactual *ex interim* expected utility maximization problem has a solution), 7 (Weakly increasing strategy is used), and 8 (Monotone effect of counterfactual beliefs on utility), Equation 13 is valid. Under Assumption 7 (Weakly increasing strategy is used), given that $z_i' < a_i(\theta_i) < z_i''$ are all used in the data, all elements of $\Theta_i(z_i')$ are less than all elements of $\Theta_i(a_i(\theta_i))$, and all elements of $\Theta_i(a_i(\theta_i))$ are less than all elements of $\Theta_i(z_i'')$, where $\Theta_i(\cdot)$ is defined in Equation 10. In particular, $\theta_i \in \Theta_i(a_i(\theta_i))$, all elements of $\Theta_i(z_i')$ are less than θ_i, and θ_i is less than all elements of $\Theta_i(z_i'')$. Then, combining Equations 11 and 12 with Equation 13, Equation 14 is valid. Equations 15, 16, 2, and 3 follow immediately, using Assumption 9 (Known bounds on valuations).

Now, for a given a_i, consider any $\tilde{\theta}_i < \Phi_{Li}(a_i')$ with $a_i' \leq a_i, a_i' \in \mathcal{A}_i^d$. If θ_i' is any valuation consistent with using action a_i', then $\theta_i' \geq \Phi_{Li}(a_i')$. Moreover, since $a_i' \in \mathcal{A}_i^d$ by construction, there is indeed some valuation θ_i' that uses action a_i'. By Assumption 7 (Weakly increasing strategy is used), the action used by valuation $\tilde{\theta}_i$ is weakly less than the action used by valuation $\theta_i' \geq \Phi_{Li}(a_i') > \tilde{\theta}_i$, so the action used by valuation $\tilde{\theta}_i$ is weakly less than a_i'. Moreover, since $\tilde{\theta}_i \not\geq \Phi_{Li}(a_i')$ by construction, valuation $\tilde{\theta}_i$ cannot use action a_i'. Consequently, player i with valuation $\tilde{\theta}_i$ must use an action strictly less than a_i'. By the contrapositive, any equilibrium action weakly greater than a_i' must correspond to a valuation weakly greater than $\Phi_{Li}(a_i')$. Consequently, because $a_i' \leq a_i$, the valuation θ_i corresponding to the use of equilibrium action a_i must be weakly greater than $\Phi_{Li}(a_i')$. Since the above holds for any $a_i' \leq a_i, a_i' \in \mathcal{A}_i^d$, the valuation θ_i corresponding to the use of equilibrium action a_i must be weakly greater than $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$. Consequently, $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ is a lower bound for the valuation corresponding to a_i. Similarly, $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$ is an upper bound for the valuation corresponding to a_i.

Therefore, considering the joint distribution of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and corresponding observed actions $A = (A_1, A_2, \ldots, A_N)$, it holds for all realizations that, for each $i \in \{1, 2, \ldots, N\}$, $\Upsilon_{Li}(A_i) \leq \theta_i \leq \Upsilon_{Ui}(A_i)$. Consequently, the partial identification result in the usual multivariate stochastic order follows from Shaked and Shanthikumar (2007, Theorem 6.B.1).

Independent valuations *Under Assumption 2, the following adjustments are made to the proof. Under Assumption 2, Equation 8 need not condition on θ_i since beliefs do not depend on valuation. Similarly, Equation 9 need not condition on θ_i. Thus, Equations 5 and 6 are valid bounds for the valuation, even without Assumptions 6 (Counterfactual* ex interim *expected utility maximization problem has a solution) and 8 (Monotone effect of counterfactual beliefs on utility). Then, by arguments similar to those used previously in the proof of Theorem 1, the valuation corresponding to*

a_i must be between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Ui}(a_i')$. Thus, the valuation corresponding to a_i must be between $\Gamma_{Li}(a_i)$ and $\Gamma_{Ui}(a_i)$ defined in Equation 7.

To establish game-structure identification of differences, $E_P(X_i|A_i = a_i) = E_P(\widetilde{x}_i(A_i, A_{-i})|A_i = a_i) = E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i) = E_P(\overline{x}_i(a_i, A_{-i}))$, where the first equality holds by definition of the game (and resulting allocations), the second equality holds by standard properties of conditioning and the law of iterated expectations (with respect to any randomness in the allocation), and the third equality holds because the actions of different players are independent. It is similar for $E_P(T_i|A_i = a_i) = E_P(\overline{t}_i(a_i, A_{-i}))$. ★ □

Proof of Theorem 2. Let $\Gamma_i(\cdot)$ defined on $\mathcal{A}_i^d$ be a strictly increasing function such that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$. At least one such $\Gamma_i(\cdot)$ does exist. Specifically, under the true data generating process, per Assumption 7 (Weakly increasing strategy is used) there are weakly increasing strategies $a_i(\theta_i)$ that generate the data. Then, let $\Gamma_i(a_i) = \text{midpoint}\{\theta_i : a_i(\theta_i) = a_i\}$ defined on $a_i \in \mathcal{A}_i^d$, the midpoint of the interval of valuations that uses action a_i per the discussion of Assumption 7 (Weakly increasing strategy is used). By definition, it must be that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$ on $\mathcal{A}_i^d$. Consider any pair $a_i \in \mathcal{A}_i^d$ and $a_i' \in \mathcal{A}_i^d$. The corresponding valuations $\theta_i' = \Gamma_i(a_i')$ and $\theta_i = \Gamma_i(a_i)$ are such that valuation θ_i' uses action a_i' and valuation θ_i uses action a_i. Since $a_i \neq a_i'$, it must be that $\theta_i \neq \theta_i'$ and by Assumption 7 (Weakly increasing strategy is used), it must be that $\theta_i \leq \theta_i'$. Consequently, it must be that $\Gamma_i(a_i) = \theta_i < \theta_i' = \Gamma_i(a_i')$, so $\Gamma_i(\cdot)$ is strictly increasing on $\mathcal{A}_i^d$.

Consider the distribution of actions according to conjectured strategies $\Gamma_i^{-1}(\cdot)$ defined on the support of $\Gamma_i(A_i)$ where $A_i \sim P(A)$. Since $\Gamma_i(\cdot)$ is strictly increasing on $\mathcal{A}_i^d$, $\Gamma_i^{-1}(\cdot)$ is strictly increasing on the support of $\Gamma_i(A_i)$. The distribution of actions is therefore $(\Gamma_1^{-1}(\Gamma_1(A_1)), \Gamma_2^{-1}(\Gamma_2(A_2)), \ldots, \Gamma_N^{-1}(\Gamma_N(A_N))) = (A_1, A_2, \ldots, A_N)$.

Now consider a realization from the distribution of valuations that is $(\Gamma_1(a_1), \Gamma_2(a_2), \ldots, \Gamma_N(a_N))$ for some $a \in \mathcal{A}^d$, which by construction uses the action a in the conjectured equilibrium. For each player i, the utility maximization problem is to maximize $\Gamma_i(a_i)E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i}))$. Specifying the player to have correct beliefs, whereby $\Pi_i(a_{-i}) = P(A_{-i})$ given the distribution of actions is the same as in the real data by the above, this is the same as maximizing $\Gamma_i(a_i)E_P(\overline{x}_i(z_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))$. The action that this valuation actually uses would satisfy the condition of utility maximization exactly when $\Gamma_i(a_i)E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{t}_i(a_i, A_{-i})) \geq \Gamma_i(a_i)E_P(\overline{x}_i(z_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))$ for all $z_i \in \mathcal{A}_i$. For $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) > 0\}$, $\Gamma_i(a_i) \geq \Gamma_{Li}(a_i) \geq \frac{E_P(\overline{t}_i(a_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))}$

by Equations 5, 6, and 7. Consequently, after re-arranging that inequality, the utility from action a_i weakly exceeds the utility from action z_i. For $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) < 0\}$, $\Gamma_i(a_i) \leq \Gamma_{Ui}(a_i) \leq \frac{E_P(\bar{t}_i(a_i,A_{-i}))-E_P(\bar{t}_i(z_i,A_{-i}))}{E_P(\overline{x}_i(a_i,A_{-i}))-E_P(\overline{x}_i(z_i,A_{-i}))}$ by Equations 5, 6, and 7. Consequently, after re-arranging that inequality, the utility from action a_i weakly exceeds the utility from action z_i. For $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) = 0\}$, since a_i is used in the data and therefore maximizes utility for some valuation, it must be that $-E_P(\bar{t}_i(a_i, A_{-i})) \geq -E_P(\bar{t}_i(z_i, A_{-i}))$, which also suffices for the utility from action a_i to weakly exceed the utility from action z_i when the valuation in particular is $\Gamma_i(a_i)$.

For the last part of the result, let $\Gamma_i(\cdot)$ defined on $\mathcal{A}_i^d$ be a strictly increasing function such that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$. Per above, at least one such function exists. Then let $\tilde{\Gamma}_i(\cdot) = \alpha\Gamma_i(\cdot) + (1-\alpha)\Gamma_{Li}(\cdot)$ for some $\alpha \in (0,1)$. Clearly, $\Gamma_{Li}(\cdot) \leq \tilde{\Gamma}_i(\cdot) \leq \Gamma_{Ui}(\cdot)$. Moreover, clearly $\tilde{\Gamma}_i(\cdot)$ is strictly increasing because $\Gamma_i(\cdot)$ is strictly increasing and $\Gamma_{Li}(\cdot)$ is weakly increasing. Further, $0 \leq \tilde{\Gamma}_i(\cdot) - \Gamma_{Li}(\cdot) = \alpha\left(\Gamma_i(\cdot) - \Gamma_{Li}(\cdot)\right) \leq \alpha\left(\Theta_{Ui} - \Theta_{Li}\right)$, so $\sup_{a_i \in \mathcal{A}_i}\left(\tilde{\Gamma}_i(a_i) - \Gamma_{Li}(a_i)\right) < \epsilon$ by taking $\alpha < \frac{\epsilon}{\Theta_{Ui}-\Theta_{Li}}$ small enough. Similar arguments based on $\tilde{\Gamma}_i(\cdot) = \alpha\Gamma_i(\cdot) + (1-\alpha)\Gamma_{Ui}(\cdot)$ establish that $0 \leq \sup_{a_i \in \mathcal{A}_i}\left(\Gamma_{Ui}(a_i) - \tilde{\Gamma}_i(a_i)\right) < \epsilon$. $\square$

Proof of Theorem 3. From Assumptions 10 (Continuous action space and no point masses in distribution of actions), 12 (Differentiable *ex interim* expected allocation and transfer), 13 (Game-structure identification of derivatives), and 14 (Non-zero marginal expected allocation), let $\mathcal{E}_i = (\text{int}(\mathcal{A}_i))^C \cup \mathcal{E}_{i,d} \cup \mathcal{E}_{i,r} \cup \mathcal{E}_{i,m}$ and $\mathcal{E} = \prod_i \mathcal{E}_i$. It follows that $P(A \in \mathcal{E}) = 0$. Then $P(\theta \in B) = P(\theta \in B, A \in \mathcal{E}^C) + P(\theta \in B, A \in \mathcal{E}) = P(\theta \in B, A \in \mathcal{E}^C) = P(\theta \in B | A \in \mathcal{E}^C)$ for any Borel set B, so it is enough to restrict the identification problem to recovering the distribution of θ from actions in $\mathcal{E}^C$. By Assumptions 3 (Action space is ordered), 4 (Optimal strategy is used), 10 (Continuous action space and no point masses in distribution of actions), and 12 (Differentiable *ex interim* expected allocation and transfer), Equation 22 is the necessary condition for any action used by player i in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C$:

$$\theta_i \left.\frac{\partial E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=\tilde{a}_i(\theta_i)} - \left.\frac{\partial E_{\Pi_i}(\bar{t}_i(a_i, a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=\tilde{a}_i(\theta_i)} = 0. \tag{22}$$

By Assumptions 1 (Dependent valuations), 7 (Weakly increasing strategy is used), 10 (Continuous action space and no point masses in distribution of actions), and 11 (Smooth distribution of valuations), conditioning on θ_i is equivalent to conditioning on $A_i = a_i(\theta_i)$, because if two distinct valuations use the same action the entire interval between those valuations would also use the same action, resulting in a point mass in the distribution of actions by Assumption 11 (Smooth distribution of valuations). So by Assumption 5 (Correct beliefs), Equation 23 is valid for actions in $\mathcal{A}_i^d \cap \operatorname{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C$:

$$\theta_i \left. \frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} - \left. \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} = 0. \tag{23}$$

Under Assumption 14 (Non-zero marginal expected allocation), Equation 24 is valid for all actions used by player i in $\mathcal{A}_i^d \cap \operatorname{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C$:

$$\theta_i = \Psi_i(A_i). \tag{24}$$

By Assumption 13 (Game-structure identification of derivatives), $\Psi_i(a_i)$ is point identified for all $a_i \in \mathcal{A}_i^d \cap \operatorname{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C \cap \mathcal{E}_{i,r}^C$. Therefore, the identification result obtains.

Independent valuations *Under Assumption 2, the following adjustments are made to the proof. Equation 22 need not condition on θ_i since beliefs are independent of valuation. Similarly, Equation 23 is valid without conditioning on A_i.* ★ □

Appendix D. On the role of equilibrium assumptions

Bayes Nash equilibrium requires that all players act rationally given beliefs (Assumption 4) and have correct beliefs (Assumption 5), so that each player chooses an action that is a best response to the distribution of actions of the other players. This assumption of equilibrium is completely standard, since it is reasonable in very many settings, but in some settings it may be too strong.[12] In the context of auction models, for example, it might be that some "novice" bidders have incorrect beliefs about the other bidders, whereas "experienced" bidders might have correct beliefs about the other bidders. Similarly, it might be that the "novice" bidders do not have sufficient understanding or experience with the auction format to bid the optimal amount given their beliefs, whereas "experienced" bidders might have that sufficient understanding and experience to bid the optimal amount given their beliefs. The difference between "novice" and "experienced" might be due to learning from participating in previous auctions, or some other reason that is observable by the econometrician, so that the

[12] Identification in games relaxing the assumption of equilibrium, or related questions, has been considered in Aradillas-López and Tamer (2008), Haile, Hortaçsu, and Kosenok (2008), Kline and Tamer (2012), Kline (2015, 2018), Syrgkanis, Tamer, and Ziani (2018), and Magnolfi and Roncoroni (2023). Kline (2018) includes a discussion of the tradeoffs between equilibrium assumptions and assumptions on the data, for identification in settings like entry games. See Maskin (2011) for a commentary on Nash equilibrium.

econometrician can distinguish which players are "novices" and which players are "experienced." For example, in electricity markets with data that includes typically unobserved valuations, which makes it possible to test bidder optimality, Hortaçsu and Puller (2008) find that "large" firms are more strategically sophisticated than "small" firms.

It is an immediate consequence of the identification strategy that it is possible to conduct exactly the same identification analysis for any player that has correct beliefs and acts rationally given those correct beliefs, regardless of whether other players have correct beliefs and/or act rationally given those beliefs. When relaxing the assumption of equilibrium, only a specific player i is assumed to have correct beliefs and act rationally given those beliefs. Assumptions that apply to all players are replaced by similar assumptions that apply only to a particular player i. For example, player i might have correct beliefs that the other players are "irrational." Then, the identification result is the "player i part" of Theorem 1, both in terms of assumptions and result. Moreover, assuming that players i and j both satisfy the assumptions, it is possible to formulate the "player i and j part" of the identification results, establishing identification of the *joint* distribution of their valuations.

If it were assumed that all players draw valuations from the same marginal distribution (i.e., "*symmetric private values*"), then identification of one player's marginal distribution of valuations is sufficient to identify all players' marginal distributions of valuations. If it were further assumed that player valuations are independent (i.e., "*symmetric independent private values*"), then identification of one player's marginal distribution of valuations is sufficient to identify the *joint distribution* of all players' distributions of valuations. Of course, it may be implausible to assume that only some players have correct beliefs and act rationally given those beliefs, while also assuming that all players draw valuations from the same marginal distribution. However, if for example all players have the same marginal distribution of valuations, but some players just happen to have more "experience" with the game for reasons unrelated to their valuation, those simultaneous assumptions may be plausible.

Appendix E. Sufficient conditions for game-structure identification of derivatives

This appendix provides one sufficient condition for game-structure identification of derivatives used in Appendix A, formalizing the idea that game-structure identification of derivatives follows from standard results on identification and estimation of conditional expectations. It is clear that other sufficient conditions also exist.

Lemma 3 (Sufficient conditions for game-structure identification of derivatives)**.** *Suppose that Assumptions 1 (Dependent valuations) and 10 (Continuous action space and no point masses in distribution of actions) are satisfied. Let an action $a_i \in \mathcal{A}_i$ be given, and suppose that the following conditions are true. It holds that $a_i \in \mathcal{A}_i^d$, and there is a set $\mathcal{S}$ containing a_i such that $\mathcal{A}_i^d \cap \mathcal{S}$ is a non-degenerate interval and such that the econometrician can point identify the conditional expectations $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ for all $a_i' \in \mathcal{A}_i^d \cap \mathcal{S}$ and $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$, where $\tilde{\mathcal{A}}_{-i}^d(a_i')$ has probability 1 according to the distribution $A_{-i}|(A_i = a_i)$. The distribution $A_{-i}|(A_i = a_i)$ is point identified. If $a_i \in int(\mathcal{A}_i)$, then $a_i \in int(\mathcal{A}_i^d \cap \mathcal{S})$. The data is $P(A, X, T)$. Then, whether or not $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist is point identified. Exists means, by definition, that the limit corresponding to the definition of the derivative exists. Moreover, if $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist, then there is game-structure identification of derivatives per Definition 5. Identification of $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ is constructive, and given by the existence and values of the limits corresponding to expressions in Equation 25:*

(25)
$$\Psi_i^x(z) = \left.\frac{\partial E_P(E_P(X_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \quad \text{and } \Psi_i^t(z) = \left.\frac{\partial E_P(E_P(T_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z}$$

This is based on the fact, in connection with standard results on identification and estimation of conditional expectations, that $\overline{x}_i(\cdot)$ and $\overline{t}_i(\cdot)$ are identifiable quantities based on the relationships $\overline{x}_i(a_i', a_{-i}) = E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $\overline{t}_i(a_i', a_{-i}) = E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$. For example, kernel regression estimators of conditional expectations are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable (e.g., Stone (1977), Devroye (1981), or Greblicki, Krzyzak, and Pawlak (1984)). This is also based on standard results on identification and estimation of the conditional distribution $A_{-i}|(A_i = a_i)$. For example, kernel estimators of conditional distributions are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable, and all realizations of the conditioning variable if $A_{-i}|(A_i = a_i)$ is suitably continuous in a_i (e.g., Stute (1986), Owen (1987), and Hall, Wolff, and Yao (1999)). Therefore, the most practically important part of the assumptions relates to the support of the data. The support condition requires that $a_i \in \mathcal{A}_i^d$ (in addition to $a_i \in \mathcal{A}_i$) and that there is a set $\mathcal{S}$ containing a_i such that $\mathcal{A}_i^d \cap \mathcal{S}$ is a non-degenerate interval, with $a_i \in \text{int}(\mathcal{A}_i^d \cap \mathcal{S})$ if $a_i \in \text{int}(\mathcal{A}_i)$. The support condition is used to identify the derivatives based on limits along a sequence of a_i' approaching a_i, where a_i' are taken in $\mathcal{A}_i^d \cap \mathcal{S}$. The condition that

$a_i \in \text{int}(\mathcal{A}_i^d \cap \mathcal{S})$ if $a_i \in \text{int}(\mathcal{A}_i)$ is used to guarantee that the usual two-sided derivative can be identified (to exist), when a_i is such that the two-sided derivative is relevant.

Proof of Lemma 3. The definitions of $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$ are given in Equation 17. By definition, $\overline{x}_i(a) = E(\widetilde{x}_i(a)) = E(\widetilde{x}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a) = E(\widetilde{t}_i(a)) = E(\widetilde{t}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$. Therefore, by substitution, the expressions in Equation 25 are valid. Let $a_i \in \mathcal{A}_i^d$ be given, and let $\mathcal{S}$ be given with the stated properties. Let $a_i' \in \mathcal{A}_i^d \cap \mathcal{S}$. By assumption, $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ are point identified for all a_{-i} in a probability 1 subset of the support of $A_{-i}|(A_i = a_i)$. Therefore, given that the distribution of $A_{-i}|(A_i = a_i)$ is point identified by assumption, $E_P(E_P(X_i|A_i = a_i', A_{-i})|A_i = a_i)$ and $E_P(E_P(T_i|A_i = a_i', A_{-i})|A_i = a_i)$ are point identified. Consequently, the existence and values of $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified by the existence and values of the limits corresponding to expressions in Equation 25. □

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