IDENTIFICATION OF INCOMPLETE INFORMATION ALLOCATION-TRANSFER GAMES IN MONOTONE EQUILIBRIUM

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ABSTRACT. This paper develops identification results for the distribution of valuations in a class of allocation-transfer games. These games determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The results allow dependent valuations, discrete parts of the action space, non-smoothness, and unknown (to the econometrician, prior to observing the data) details of how the allocations and transfers are determined. The identification strategy is based on the assumption of a single monotone equilibrium used in the data, in which players use strategies that are weakly increasing functions of their valuations for the object being allocated. As extensions, the identification strategy accommodates certain relaxations of the equilibrium assumption, while maintaining the assumption of the use of monotone strategies.

JEL codes: C57, D44, D82. Keywords: identification, incomplete information, monotone equilibrium.

1. INTRODUCTION

This paper develops identification results for a class of allocation-transfer games that involve allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. Each of the players has a privately-known valuation for a unit of the object, and uses a strategy that relates its valuation to the action it takes in the game. The valuations can be dependent, including but not limited to "affiliated values." The identification result concerns recovering the distribution of these valuations from the data. The data corresponds to multiple instances ("plays") of the game. Partial identification results are stated in terms of "bounds"

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on the distribution of valuations in the sense of the usual multivariate stochastic order. Examples of allocation-transfer games include contests, auctions, public good provision, and various strategic market models.

The identification strategy involves using the utility maximization problem to recover information about the unobserved valuation corresponding to an observed action. More specifically, the main identification strategy involves using an equilibrium assumption that combines the assumption of utility maximization and correct beliefs. As discussed in Remark 5, this standard assumption of equilibrium can be relaxed in various ways. Similar to many other identification results in settings involving incomplete information, the identification strategy specifically relies on the assumption of a *single* equilibrium used in the data; see Remark 1 for further discussion. Hence, the identification strategy relates to an extensive literature in econometrics that uses utility maximization as a source of identification. This approach is especially common in industrial organization, including (but not limited to) in models of firm behavior and monopoly/oligopoly (e.g., Rosse (1970), Bresnahan (1982), Lau (1982), Berry et al. (1995)) and models of auctions (e.g., Paarsch (1992), Donald and Paarsch (1993, 1996), Laffont et al. (1995), Guerre et al. (2000), Athey and Haile (2002), and Aradillas-López et al. (2013)). These literatures have been reviewed in Berry and Tamer (2006), Paarsch and Hong (2006), Athey and Haile (2007), Berry and Reiss (2007), Reiss and Wolak (2007), Kline et al. (2021), and Kline and Tamer (2023) among other places.

In addition to assuming equilibrium, the identification results assume *monotone* equilibrium. Each player uses a strategy that expresses its action as a function of its valuation. In a monotone equilibrium, the strategies are weakly increasing functions. In a monotone equilibrium, if the valuation of a player increases then that player puts forth more effort in contest models, bids more in auction models, offers/demands more in market models, or contributes more in public good provision models. The monotone equilibrium assumption is plausible, particularly in view of the economic theory literature that has many results establishing sufficient conditions for existence of monotone equilibrium; see Section 3.2.

There are two distinct parts to the "plausibility" of the assumption of monotone equilibrium. First, it is plausible that the monotone equilibrium is *selected* in the case of multiple equilibria, when a monotone equilibrium exists. Second, it is plausible that a monotone equilibrium *exists*. This is because the economic theory literature establishes existence of a monotone equilibrium under a plausible set of sufficient conditions on model primitives, as mentioned later in the paper, surrounding the discussion of Assumption 5.

It is important that the assumption concerns *weakly* increasing strategies rather than *strictly* increasing strategies. This is consistent with the results of the economic theory literature. *Weakly* increasing strategies allows for the possibility that multiple valuations "pool" on the same action, resulting in a "flat spot" in the strategy. The issue of "pooling" is directly related to the notion of a "semiseparating" or "semipooling" equilibrium. The assumption of *strictly* increasing is too strong in games involving discrete actions. In those games, a range of valuations will generally use any particular discrete action. More generally, it is also too strong in other games where the strategies involve "flat spots," even with continuous actions, as illustrated in Example 1, Example 2, Example 5, and Example 6. As noted below, the identification strategy *does not* require smoothness conditions on the players' strategies.

Strictly increasing strategies are invertible but *weakly* increasing strategies are not. This difference between the assumption of a *strictly* increasing strategy and the assumption of a *weakly* increasing strategy has a substantial impact on the identification strategy. The point identification result (Theorem 6) requires *strictly* increasing strategies, among other conditions.

The identification results in this paper have multiple features. Note that these features *interact* in important ways, as discussed below.

The most obvious feature is the identification result applies to a class of allocation-transfer games that involve allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players.[1] This class includes models of contests, auctions, procurement auctions and related models of oligopoly competition, partnership dissolution, public good provision, and strategic (non-"price taking") markets. The possible interpretations of the actions include effort in contest models, bids in auction models, bids/asks in market models, or contributions in public good provision models. In some games, as in auctions of a single unit, at most one player can

[1]Larsen and Zhang (2018) (in work contemporaneous to the first version of this paper) conduct an identification analysis in a similar set of models. However, they take a different approach: in particular, they do not use the monotone equilibrium assumption, and they focus on the case of independent valuations. Even with independent valuations, by construction, the use of the monotone equilibrium assumption can tighten the identified set relative to not using that assumption. In their setting, they are able to deal with important issues including limited observability of actions and unobserved game-level heterogeneity. The brief discussion of dependent valuations requires invertible strategies (see their page 30); as emphasized in this introduction, dealing with non-invertible strategies is a central part of this paper (e.g., since it allows discrete action spaces and/or flat spots in the strategy). Overall, as also noted in Larsen and Zhang (2018), despite working in a similar class of models, the actual identification analyses in these two papers are quite different and complementary.

be allocated a unit of the object. In other games, as in auctions of multiple units or public good provision, multiple players can be allocated a unit of the object. In some games, as in contests, the allocation can be non-deterministic. Therefore, the identification result can be viewed as exploring the identification power of the assumption of the use of monotone strategies across this entire class of allocation-transfer games.

The next feature of the identification results concerns the *generality of* the assumptions used by the identification strategy. The identification strategy allows for dependent valuations. And, the identification strategy allows for "flat spots" in the strategy, discrete parts of the action space, and non-smoothness (discussed further below). The action space can be discrete, continuous, or combinations of discrete and continuous. With "flat spots" (including those caused by discrete actions), a range of valuations use the same action, so those valuations cannot be distinguished based on observed behavior. This should be expected to result in partial identification. Further, the utility maximization problem depends on the beliefs held by the player, which depend on the valuation of the player. The beliefs of players with different valuations are generally distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

Although the literature on incomplete information games has focused on independent unobservables, there are existing results for cases of dependent unobservables in specific models. Li et al. (2000), Li et al. (2002), and Campo et al. (2003) study the case of first-price sealed-bid auctions; Aradillas-López et al. (2013) study the case of ascending auctions. Aradillas-Lopez (2010), Wan and Xu (2014), Xu (2014), and Liu et al. (2017) study the case of binary (entry) games. Some of those papers also use the assumption of a monotone Bayesian Nash equilibrium in their settings, in different ways from the use in this paper. Li et al. (2000), Li et al. (2002), and Campo et al. (2003) have a continuous action space, and use the assumption of a *strictly* increasing strategy together with smoothness (differentiability) conditions. Wan and Xu (2014), Xu (2014), and Liu et al. (2017) have a binary action space, and use the assumption in order to focus the identification strategy on a setup involving a binary variable that has a certain threshold-crossing form. The identification strategy in this paper concerns the assumption of a *weakly* increasing strategy in a general action space, which is not necessarily as rich as a continuous action space, and not necessarily as coarse as a binary action space. The partial identification results in this paper do not rely on smoothness conditions. Also importantly, beyond the issue of the cardinality of the action space, the games considered by the different identification strategies also differ.

In particular, despite the shared issue of "discrete actions," the models and identification strategy considered in this paper are distinct from those considered in the literature on the "econometrics of (entry) games" (e.g., Tamer (2003), Aradillas-López and Tamer (2008), Ciliberto and Tamer (2009), Aradillas-Lopez (2010), Bajari et al. (2010a), Bajari et al. (2010b), de Paula and Tang (2012), Kline and Tamer (2012, 2016), de Paula (2013), Wan and Xu (2014), Xu (2014), Kline (2015, 2016), Liu et al. (2017), Aradillas-López (2020), Ciliberto et al. (2021)). In models of market entry, the model primitive for each player is a utility function that depends on the actions of all market participants. In allocation-transfer games, the model primitive for each player is a valuation. Consequently, the two sets of models have a different payoff structure and interaction structure, and so allocation-transfer games are different from models of market entry, and so the corresponding identification strategies are also different. Furthermore, in models of market entry, the identification strategy generally concerns characterizing (often parametric) dependence of utility on payoff shifters (including the strategic impact of other entry decisions). In allocation-transfer games, the object of interest is the (non-parametric) distribution of valuations. Additionally, observed payoff shifters with exclusion restrictions are not used in the identification strategy in this paper, whereas that is central in the market entry game literature. Even within the class of entry game models, allowing for discrete but non-binary action spaces adds additional complications (e.g., Aradillas-López and Gandhi (2016) and Aradillas-López and Rosen (2022)); in particular, these existing results do not accommodate incomplete information with dependent unobservables (following the standard in the entry game literature). However, the identification strategy in this paper does apply to some models of strategic (non-"price taking") market behavior, which can describe behavior *after* entry into a market, as in Example 6.

Discrete actions are common in empirical practice. For example, when the action is a monetary amount (e.g., a "bid" in an auction or "contribution" in public good provision), almost any realistic implementation in practice will place restrictions on the allowed bids. For instance, the implementation might require bids that are an integer multiple of some fixed amount (e.g., the allowed bids might be 5 dollars, 10 dollars, 15 dollars, etc.).[2] Discrete actions can also arise for other reasons. For instance,

[2] "Discrete" can be used with different definitions, which are worth distinguishing. Hortaçsu and McAdams (2010) studies an identification problem (and empirical application) in discriminatory price divisible goods auctions with

some public good provision models have a binary action space: contribute or not contribute, as in Example 5. Allowing discrete actions also accommodates fundamentally "non-numerical" actions, for example a binary "participation" decision when "participation" in the game is voluntary, as in some auction models in Example 2.

Another feature of the main partial identification strategy is that it accommodates various sources of non-smoothness. Relative to conditions on the players' strategies, the partial identification strategy accommodates discontinuous (and hence not differentiable) strategies. There are many games in which equilibrium exists in "step function" strategies, which are not even continuous. See for instance Example 5 and Example 6. This can happen even with a continuous action space. Relative to conditions on the distribution of valuations, the partial identification strategy accommodates point masses. For example, there might in particular be a point mass in the distribution of valuations at 0.[3]

Overall, the main partial identification strategy does not involve derivatives. This is a difference from many of the identification results in the broader literature that are based on smoothness conditions and the first order condition approach to utility maximization. The limiting point identification result in Appendix A does involve smoothness conditions and derivatives.

A further feature of the identification strategy is the identification strategy does not depend on the econometrician having *ex ante* (prior to observing the data) knowledge of the details of how the allocations and transfers are determined on the basis of the actions of the players, because it is possible to use the data to identify these objects. For example, the econometrician does not need to *ex ante* know the "contest success function" in models of contests, which relate the effort put forth by the players to the probabilities that each of them win the contest, as in Example 1. For another

independent private values. Kastl (2011) studies an identification problem (and empirical application) in uniform price divisible good auctions with (mainly) independent private values. In those models, bidders submit a bid function that specifies a quantity demanded for each possible price. Hence, neither model is covered by the allocation-transfer game framework studied in this paper, because those models deal with an action space that is a bid function rather than just a scalar bid. More importantly, the notion of "discrete" action is also different. In particular, Kastl (2011) uses "discrete" (per Kastl (2011, Assumption 3)) as a statement about the step function nature of the bid functions, where each player submits a bid function that is a step function, and therefore characterizable by a discrete vector of prices and quantities that characterize each "step" of the bid function. Hortaçsu and McAdams (2010) similarly emphasize step bid functions. However, the actual price and quantities at each step of the bid function is unrestricted. By contrast, as applied to auctions, this paper uses discrete as a statement on the restriction of the allowed bid levels. So, the players can only bid, for example, integer multiples of some minimal bid level. An earlier version of Hortaçsu (2002) looked at a model with a discrete grid of possible prices, and hence with a "discrete" action space more similar to the discreteness in this paper. Of course, the overall identification problem (and hence identification strategy) is still different from the identification problem addressed in this paper, particularly given the differences in the models being identified. The identification strategy in this paper does not restrict to auctions or independent values.

[3]The related economic theory literature often accounts for this, for example in Harris and Raviv (1981), Maskin and Riley (2000a, Section 3), Maskin and Riley (2000b, Section 4), Maskin and Riley (2003), Lebrun (2006, Section 3), Lovejoy (2006), and Kaplan and Zamir (2012), among other examples.

example, the econometrician does not need to *ex ante* know the endogenous quantity function in auctions where the quantity of the object allocated depends on the actions of the players, as in a "supply curve," as in Example 2. Such features of the game can be identified from the data, rather than assumed *ex ante* known.

This feature of the result may suggest complications in certain counterfactual analyses, since it may suggest that the econometrician may not be able to generate the outcomes predicted by the model for a given specification of the model primitives (the distribution of valuations). However, under certain conditions, the econometrician can fully point identify how allocation and transfers are determined, as in Section 4.2. Even if those conditions fail to hold, the econometrician may conduct counterfactual analyses that involve directly specifying an interesting (possibly counterfactual) model of how allocations and transfers are determined.

The results focus on using the assumption of Bayesian Nash equilibrium. Remark 5 discusses the fact that the results can accommodate certain relaxations of the equilibrium assumption, another feature of the identification results.

The remainder of the paper is organized as follows. Section 2 sets up the allocation-transfer game framework studied in this paper. Section 3 discusses the assumptions. Section 4 provides the identification results. Section 5 provides a numerical illustration. Finally, Section 6 concludes. Appendix A provides sufficient conditions for point identification, relating to a discussion in Section 4.6 about the "limit" when the action space becomes an interval alongside other conditions (including the use of *strictly* increasing strategies and smoothness and differentiability conditions). Appendix B provides examples of the allocation-transfer games framework studied in this paper. Appendix C collects the proofs.

2. Allocation-transfer game framework

There are $N \geq 2$ players in the game. Players are indexed by $i = 1, 2, \ldots, N$. In principle, the results could apply to some "single-player games" with $N = 1$, if the assumptions hold in such a game, but the focus is on multiple-player games. As illustrated via specific examples in Appendix B, many economic environments can be modeled using this allocation-transfer game framework. This includes contests, auctions, procurement auctions and related models of oligopoly competition, partnership dissolution, public good provision, and strategic (non-"price taking") market behavior.

The set of all player indices is $\mathcal{I} = \{1, 2, \ldots, N\}$. The identification analysis allows for the possibility of assuming that only specific players satisfy the assumptions of "maximizing utility given correct beliefs;" these players will be the index set $\mathcal{J} = \{1, 2, \ldots, N_1\}$ where $N_1 \leq N$. In a standard application that assumes Bayesian Nash equilibrium, $N_1 = N$. It is without loss of generality that $\mathcal{J}$ is the first N_1 player indices, by re-labeling player indices if necessary.

2.1. Utility functions. Player i has valuation θ_i for a unit of the object. The utility of player i with valuation θ_i, and who receives allocation x_i of the object and transfers away ("pays") t_i units of money is

$$U(\theta_i, x_i, t_i) \equiv \theta_i x_i - t_i.$$

The sign of t_i is unrestricted, so player i can be "paid" if t_i is negative. The allocation and transfers are determined by the game, described shortly in Section 2.3. For example, the monetary transfer could be the payment in an auction model, the "price" in a market model, or the contribution in a public good provision model. This utility function is standard in the economic theory literature.

Assumption 1 (Dependent valuations)**.** *It is common knowledge among the players that* $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_N)$ *is drawn from* $F(\theta)$*, and* θ_i *is the private information of player* i*.*

This main assumption on the distribution of valuations is standard. The econometrician need not know the support of θ. It is allowed that θ is continuous, discrete, or some combination. The identification analysis can also be conducted under the further assumption of independent valuations:

Assumption 1* (Independent valuations)**.** In addition to Assumption 1, player valuations are independent, in the sense that the components of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ are independent random variables, so $F(\theta) = F_1(\theta_1)F_2(\theta_2)\cdots F_N(\theta_N)$.

Independent valuations is treated as a special case, because it is often treated as a special case in the related literature. Thus, a reader may wonder what impact independent valuations would have on the identification strategy in this paper. It turns out that imposing Assumption 1* has relatively little impact on the identified bounds; rather, it simplifies the functional form of the identified bounds, and eliminates the need to make certain assumptions to justify the bounds. See Remark 6.

It is not assumed that different players draw their valuations from the same marginal distribution, as $F_i(\cdot)$ need not equal $F_j(\cdot)$, which is useful for example to model "weak" and "strong" bidders in auctions or asymmetries between buyers and sellers in models of market behavior.

2.2. Actions. After realizing θ_i, player i takes an action a_i from the action space $\mathcal{A}_i$. The interpretation of actions depends on the game, and includes efforts in contest models, bids in auction models, announcements (bids/asks) in market models, and contributions in public good provision models.

For "monotonicity" of a strategy to be a well-defined concept, it is necessary that $\mathcal{A}_i$ is ordered. This is accomplished by assuming that $\mathcal{A}_i$ can be encoded to be a subset of real numbers.

Assumption 2 (Action space is ordered)**.** *For each $i \in \mathcal{I}$, the econometrician knows the action space for player i is $\mathcal{A}_i \subseteq \mathbb{R}$.*

As a subset of $\mathbb{R}$, $\mathcal{A}_i$ inherits the ordering of the real numbers, and $\mathcal{A}_i$ can be continuous, discrete, or some combination of continuous and discrete.

There is not necessarily a "numerical interpretation" of the actions in $\mathcal{A}_i$, similar to how the numerical encodings in categorical choice models may or may not have a substantive "numerical interpretation." For example, in games with voluntary participation including auctions with participation costs, one of the actions is the action "DNP" for "do not participate." The numerical encoding of "special" actions as numbers in $\mathcal{A}_i$ respects the ordering of the actions. For example, in auctions with voluntary participation, generically players with low valuations choose to not participate, so it makes sense to define DNP to be the lowest possible action, in order for the equilibrium strategy to be monotone. It could be that DNP is encoded as -1 or -2, for example. The specific numerical encoding is irrelevant.

2.3. Allocations and transfers. The vector of all players' actions is $a = (a_1, a_2, \ldots, a_N)$, the vector of all players' allocations is $x = (x_1, x_2, \ldots, x_N)$, and the vector of all players' transfers is $t = (t_1, t_2, \ldots, t_N)$. That is, lower-case quantities are from the point of view of the players.

The game determines the allocations and transfers based on the actions taken by the players. Even for a given profile of actions, non-deterministic allocations and transfers are allowed, for example to allow "noise" in the process of determining a winner in a contest, as in Example 1. On the basis of all players' actions a, the realized allocation and transfer is a realization[4] from the joint distribution of

$$(\tilde{x}(a), \tilde{t}(a)) = (\tilde{x}_1(a), \tilde{x}_2(a), \ldots, \tilde{x}_N(a), \tilde{t}_1(a), \tilde{t}_2(a), \ldots, \tilde{t}_N(a)),$$

where $\tilde{x}_i(a)$ (resp., $\tilde{t}_i(a)$) is a random variable that characterizes the distribution of allocations (resp., transfers) for player i given that the players take actions a. These distributions characterizing the allocations and transfers are part of the specification of the game rules.

If $(\tilde{x}_1(a), \tilde{x}_2(a), \ldots, \tilde{x}_N(a), \tilde{t}_1(a), \tilde{t}_2(a), \ldots, \tilde{t}_N(a))$ is a degenerate random variable, then the allocation and transfer is deterministic when the players take actions a. As a function of all players' actions, the *expected* allocation to player i is $\bar{x}_i(a) = E(\tilde{x}_i(a))$ and the *expected* transfer from player i is $\bar{t}_i(a) = E(\tilde{t}_i(a))$. Under the assumptions of the identification analysis, only *expected* allocation and *expected* transfer matters. However, that is a (very modest) result, and involves the considerations of Footnote 4, so the setup begins with the specification of the distribution of allocations and transfers.

As is standard in the literature, the players know the distributions of $(\tilde{x}(\cdot), \tilde{t}(\cdot))$. In other words, the players know the "rules" of the game.

Conversely, the identification results apply regardless of whether or not the econometrician *ex ante* knows (before observing the data) the distributions of $(\tilde{x}(\cdot), \tilde{t}(\cdot))$, and/or the *expected* allocations and transfers $(\bar{x}(\cdot), \bar{t}(\cdot))$. In particular, any "randomness" that underlies non-deterministic allocations and transfers need not be explicitly modeled or *ex ante* known by the econometrician. If the econometrician does not *ex ante* know these objects, then it is possible to use the data to identify these objects.

The following two examples are selected from Appendix B. Both are further illustrated via a numerical illustration of the identification results in Section 5.

Example 1 (Contests)**.** In contests, the actions are "effort" toward winning a valuable object. $\bar{x}_i(a)$ is the "contest success function" that gives the probability that player i wins given the efforts of all players. Common examples are provided in Example 1 in Appendix B. It is plausible that the econometrician does not have *ex ante* knowledge of the contest success function. And, $\bar{t}_i(a)$ is the

[4] By construction, these realizations are draws from the joint distribution and therefore by construction are independent of all other model quantities (e.g., the valuations of the players). This condition formalizes the notion that the allocation and transfer "don't depend on" anything except the actions of the players, and is (often implicitly) a standard condition in the related economic theory literature. Of course, the realized allocation and transfer will *indirectly* depend on the players' valuations, since the players' valuations determine the players' actions and the players' actions determine the realized allocation and transfer. For example, in the case of a tie for high bid in an auction, the auctioneer could flip a coin to determine who wins, but the outcome of the coin flip cannot somehow be "correlated" with the valuations of the players.

transfer from player i. Generally in contest models, at least the winning player transfers its "effort" and potentially other losing players transfer at least some fraction of their "effort."

Example 2 (Auctions)**.** The transfer rule in an auction varies substantially across auction formats. For example, in a standard nth-price auction, $\bar{t}_i(a)$ is specified so that a winner of the auction pays the n-th highest bid. For the allocation rule, a common property is that the bidder that places the highest bid wins the auction and is allocated the object, subject to complications like reserve prices or tie-breaking rules. The auction might involve multiple units, in which case the corresponding number of highest bidders are all allocated a unit of the object, possibly with corresponding adjustments to the transfer rule. Example 2 in Appendix B provides more discussion of the specification of auctions. Note that procurement auctions and some oligopoly models are quite similar, as discussed in Example 3 in Appendix B.

2.4. Data and identification problem. The identification problem concerns recovering the distribution of valuations from observing many instances ("plays") of the game. For context, the related literature on identification in auctions has typically considered this identification problem in the case of auctions specifically. Variables relating to the actions, allocations, and transfers in *upper-case letters* represent quantities in the data, whereas quantities in *lower-case letters* represent variables in the underlying game. For example, A_i is the realized action in the data from player i, whereas a_i is the action variable in the underlying game from player i. Therefore, from each play of the game, the realized actions are $A = (A_1, A_2, \ldots, A_N)$, the realized allocations are $X = (X_1, X_2, \ldots, X_N)$, and the realized transfers are $T = (T_1, T_2, \ldots, T_N)$. Unless otherwise stated, the econometrician observes population data on the actions, allocations, and transfers. Hence, unless otherwise stated, the population data is $P(A, X, T)$. In each instance of the game, by definition (X, T) is a draw from $(\tilde{x}(A), \tilde{t}(A)) = (\tilde{x}_1(A), \tilde{x}_2(A), \ldots, \tilde{x}_N(A), \tilde{t}_1(A), \tilde{t}_2(A), \ldots, \tilde{t}_N(A))$.

In some cases, the identification strategy can be based on less than full data on $P(A, X, T)$. Specifically, if the econometrician *ex ante* knows $(\tilde{x}(a), \tilde{t}(a))$, or at least $(\bar{x}(a), \bar{t}(a))$, then the identification strategy can be based on only $P(A)$. If the game involves a "two-part transfer," as in an auction with a participation cost, then the identification strategy can in certain cases be based on data from only one part of the transfer. See the discussion in Section 4.2.

3. Assumptions

The following sections build up to the identification result, by introducing and discussing the assumptions (Sections 3.1 to 3.3), describing the notion of partial identification in this setting (Section 4.1), presenting the identification of the "game structure" (Section 4.2), and then presenting the identification result (Section 4.3). A sketch of the identification strategy is available in Section 4.4. Sharpness is discussed in Section 4.5, and a limiting identification result is discussed in Section 4.6.

3.1. Baseline assumptions. The following baseline assumptions are used. These assumptions are standard from the economic theory literature and commonly used in econometrics, and so the discussion of them is relatively brief. The next section discusses the monotone equilibrium assumption that is the focus of this paper.

The players are assumed to be risk neutral, and therefore the *expected* allocations and transfers $\overline{x}_i(a)$ and $\overline{t}_i(a)$ determine *ex post* expected utility of player i as a function of its valuation and all players' actions:

$$\overline{U}_i(\theta_i, a) = \theta_i \overline{x}_i(a) - \overline{t}_i(a).$$

In this paper, *ex post* refers to after the realization of the actions of all players, which still can involve the expectation with respect to any randomness of the allocation rule and transfer rule. Because of risk neutrality and expected utility, the utility that is actually realized (based on actually realized allocation and transfer) plays no role distinct from *ex post* expected utility. *Ex interim* refers to before the realization of the actions of all players, but after an individual player realizes its own valuation, which involves taking the expectation with respect to the player's beliefs about the other players' actions and the randomness of the allocation rule and transfer rule.

Because player i does not know the actions of the other players when it chooses its action, it must form beliefs about the actions of the other players. Player i's beliefs are a distribution $\Pi_i(a_{-i}|\theta_i)$, defined over the actions of the other players, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, that conditions on player i's realized valuation θ_i.

Independent valuations. *Under Assumption 1* (Independent valuations), player i's beliefs are $\Pi_i(a_{-i})$, independent of player i's realized valuation.* ★

Therefore, *ex interim* expected utility of player i as a function of its valuation and its action is

$$V_i(\theta_i, a_i) = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i) - E_{\Pi_i}(\bar{t}_i(a_i, a_{-i})|\theta_i). \tag{1}$$

The valuation θ_i affects the expected allocation and expected transfer experienced by player i, even for a fixed action a_i, since player i's expected allocation and expected transfer depend on player i's beliefs about the other players' actions, and therefore on θ_i.

It is assumed that player i is rational, in the sense of using an optimal action given its beliefs.

Assumption 3 (Optimal strategy is used)**.** *For each $i \in \mathcal{J}$, for each possible valuation θ_i, player i uses a strategy $a_i(\theta_i)$ when it has valuation θ_i, with*

$$a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i)), \tag{2}$$

so each action taken according to the strategy $a_i(\theta_i)$ maximizes ex interim *expected utility.*

In this assumption and other places, "possible valuation" means a valuation that is possible according to the (unknown) distribution of valuations. Assumption 3 does not state that player i has correct beliefs. Instead, the subsequent Assumption 4 states that player i has correct beliefs. Also, Assumption 3 allows the use of a mixed strategy, but the identification strategy is based on the assumption of monotone equilibrium in monotone pure strategies, as formalized and discussed subsequently in Assumption 5. Breaking up the assumptions makes it easier to discuss the different roles of the assumptions of using an optimal strategy, correct beliefs, and monotone equilibrium.

Assumption 3 assumes that players 1 through N_1 use an optimal strategy, from the index set $\mathcal{J}$. The econometrician can specify N_1. Of course, setting $N_1 = N$ says that all players use an optimal strategy. If $N_1 < N$, then some players may not use an optimal strategy. The identification strategy accommodates the possibility that only some players use an optimal strategy; if so, then the identification result restricts to the distribution of valuations of those players. See also Remark 5.

Let $P(A, X, T, \theta)$ be the "infeasible" data, regardless of whether those variables are observed by the econometrician. Let $P(A_{-i}|\theta_i)$ be the distribution of $A_{-i} = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_N)$ conditional on the valuation θ_i of player i. Of course, θ_i is not observed by the econometrician, so the econometrician cannot condition on θ_i. In a Bayesian Nash equilibrium, each player's beliefs are correct and correspond to the actual distribution of actions of the other players.

Assumption 4 (Correct beliefs)**.** *For each $i \in \mathcal{J}$, player i has correct beliefs, in the sense that, for each possible valuation θ_i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B.*

Independent valuations. *Under Assumption 1* (Independent valuations), the assumption of correct beliefs is $\Pi_i(a_{-i} \in B) = P(A_{-i} \in B)$, since then beliefs do not depend on θ_i.* ★

As in other incomplete information game identification results, this assumption of correct beliefs implicitly assumes the realized distribution of actions (i.e., the data) comes from a single equilibrium. If multiple equilibria were used in the data, the realized distribution of actions in the data would be a mixture over the beliefs held by the player across equilibria. One trivial sufficient condition for a single equilibrium being used in the data is that a single equilibrium exists in the game. The economic theory literature has many results on equilibrium uniqueness; see Appendix B. In particular, it can be that there is a unique equilibrium that involves using monotone strategies, even if there are other equilibria that do not involve monotone strategies.

Remark 1 (Testing or relaxing the assumption of a single equilibrium)**.** It is possible to test the condition that a single equilibrium is used in the data. de Paula and Tang (2012) establish a test for the use of multiple equilibria in the data in binary incomplete information games. As summarized also in de Paula (2013), the main idea is that correlation in observed actions can arise only from the use of multiple equilibria, under the key assumption of independence of unobservables. The situation here is analogous. The distribution of observed actions from a given equilibrium, $(a_1(\theta_1), a_2(\theta_2), \ldots, a_N(\theta_N))$, has independent components in the case of independent valuations. Thus, any dependence among observed actions can be taken as evidence for multiple equilibria used in the data. As with the broader literature, it seems unclear if similar results hold in the case of dependent valuations (or more generally dependent unobservables). Related results in Aradillas-López and Gandhi (2016), Kline (2016), and Tomiyama and Otsu (2022) all also require independent unobservables.

The same considerations suggest that allowing for multiple equilibria in the data and dependence in the valuations would require some other new step in the identification strategy. While most of the literature focuses on the case of a single equilibrium used in the data, Xiao (2018), Aguirregabiria and Mira (2019), and Fan et al. (2024) focus on the issue of multiple equilibria in some incomplete information games, while ruling out dependence in the private information unobservables as a main assumption; notably, Xiao (2018) includes a discussion of the difficulty in simultaneously allowing for multiple equilibria in the data and dependence in the unobservables and Aguirregabiria and

Mira (2019, page 1694) observes that "two types of restrictions are crucial for our identification results: independence between private players' private information [...]". In fact, as standard in that part of the incomplete information game literature, those results require a known distribution of unobservables (see Xiao (2018, page 332) and Aguirregabiria and Mira (2019, Assumption 2) and Fan et al. (2024, Assumption 2.1 or 5.1)), which would obviously not be suitable for the goal of this paper of identifying the distribution of valuations. Sweeting (2009) uses the existence of multiple equilibria as a source of identification, while also assuming independence of the Type I extreme value unobservables on page 718. Grieco (2014) considers a game with "flexible information structures" allowing for multiple equilibria, while assuming independence of private information drawn from a normal distribution on page 307. Thus, similar to that literature, this paper does not contribute to the potentially interesting question of identification of incomplete information games with multiple equilibria and dependent unobservables.

Similar to Assumption 3, Assumption 4 accommodates the possibility that only some players have correct beliefs. Assumptions 3 and 4 with $N_1 = N$ entails a Bayesian Nash equilibrium.

Under correct beliefs held by player i, $V_i(\theta_i, a_i) = \theta_i E_P(\overline{x}_i(a_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i, A_{-i})|\theta_i)$.

3.2. Monotone equilibrium. The main assumption of the identification strategy is monotone equilibrium.

Assumption 5 (Weakly increasing strategy is used)**.** *It holds that:*

(a) *For each $i \in \mathcal{J}$, player i uses a pure strategy.*

(b) *For each $i \in \mathcal{J}$, player i's pure strategy $a_i(\cdot)$ is a weakly increasing*[5] *function.*

The use of pure strategies implies that $a_i(\theta_i)$ is a particular action rather than a non-degenerate distribution. Equilibrium existence in pure strategies is a general result for games with incomplete information. The general economic theory (and existence) of equilibria in pure strategies has been studied, for example, in Radner and Rosenthal (1982), Milgrom and Weber (1985), Dasgupta and Maskin (1986), Vives (1990), and Reny (1999), in addition to citations elsewhere in this paper, particularly Appendix B.

[5]It is straightforward to accommodate a weakly *decreasing* strategy, because a weakly *decreasing* strategy can be translated into a weakly *increasing* strategy by flipping the signs on the allocation rule and valuations, because if the strategy is weakly *decreasing* in the valuation θ_i, then the strategy is weakly *increasing* in the "negative valuation" $\hat{\theta}_i = -\theta_i$ with "negative allocation" $\hat{x}_i(a) = -\widetilde{x}_i(a)$. Note that $\hat{\theta}_i \hat{x}_i(a) = \theta_i \widetilde{x}_i(a)$, so utility is unaffected by flipping the signs in this way.

Results establishing general conditions for existence of equilibrium in monotone strategies include Athey (2001), McAdams (2003, 2006), Van Zandt and Vives (2007), and Reny (2011). The setup of the game in this paper can be viewed as a particular specification of the utility function relative to that literature. As a focal result in the literature, the key assumption of Athey (2001, Theorem 1) is a "single crossing condition" for incomplete information games that requires that the utility functions satisfy a "single crossing property of incremental returns" whenever all (other) players use a monotone strategy. This can be interpreted as requiring that a (marginal) increase in the action increases utility more when the valuation is (marginally) higher. Under differentiability, this can be interpreted as requiring a positive second cross-derivative of the *ex interim* expected utility function with respect to the action and the valuation. Athey (2001, Section 4.2) explores the single crossing condition in a class of games very similar to the class of allocation-transfer games studied here.

The economic theory literature has also established existence of equilibrium in monotone strategies in specific games, as cited elsewhere in this paper, particularly Appendix B. With reference to specific games, the results can be expressed even more explicitly, with perhaps more concretely intuitive interpretations. A key assumption in many results establishing Assumption 5 is affiliated valuations, which is a particular form of positive dependence among the valuations across players. It seems plausible that valuations would be positively dependent, rather than negatively dependent. Particularly in the context of affiliation in auctions, see Milgrom (2004, Section 5.4.1) for details. The identification strategy only requires Assumption 5. Equilibria in monotone strategies can exist even without affiliated valuations; see for example Monteiro and Moreira (2006).

Example 1 (Contests, continuing from p. 10)**.** Assumption 5 (Weakly increasing strategy is used) requires the plausible condition that players with higher valuations put forth more effort. This has been proven to hold under general conditions, as detailed in Example 1 in Appendix B.

Example 2 (Auctions, continuing from p. 11)**.** Assumption 5 (Weakly increasing strategy is used) requires the plausible condition that players with higher valuations place higher bids. This has been proven to hold under general conditions, as detailed in Example 2 in Appendix B.

This paper uses monotonicity differently from other common uses of monotonicity in econometrics. In other areas of econometrics, monotonicity commonly relates to the functional relationship between two observed variables, and the functional relationship is the object of interest. Monotonicity has been imposed as a shape restriction on the estimator in regression models (e.g., Mukerjee (1988), Ramsay (1988, 1998), and Mammen (1991)), and has been used in the identification of treatment effects models (e.g, Manski (1997), and Manski and Pepper (2000, 2009)). By contrast, when assuming use of monotone strategies, the monotonicity relates to the equilibrium functional relationship between the observed action and the unobserved valuation, and the distribution of the unobserved valuations is the object of interest.

Under Assumption 5, players use *weakly* increasing strategies, but not necessarily *strictly* increasing strategies. This is consistent with the results of the economic theory literature, and accommodates the possibility that player i with valuation θ_i takes the action $a_i(\theta_i)$ and player i with valuation $\theta_i' \neq \theta_i$ also takes the same action $a_i(\theta_i') = a_i(\theta_i)$. In such a case, multiple valuations "pool" on the same action. Discrete actions generically lead to such "flat spots" in the strategy. Such "flat spots" can also arise even without discreteness in the action space, for example as discussed in Example 1, Example 2, Example 5, and Example 6. This is directly related to the notion of a "semiseparating" or "semipooling" equilibrium. Sometimes, but not always, the pooling happens on the smallest and/or largest possible action, due to the associated boundary constraint in the optimization problem in Equation 2. Pooling can also happen at other actions, and for different reasons, as exemplified by the use of "step function" strategies in Example 5 and Example 6.

Under Assumption 5, the set $\Theta_i(a_i^*) = \{\theta_i : a_i(\theta_i) = a_i^*\}$ of valuations θ_i that could possibly use given action $a_i^* \in \mathcal{A}_i$ is necessarily a convex set (possibly empty or singleton). The term "convex" is used to refer to the fact that if $\theta_i' \in \Theta_i(a_i^*)$ and $\theta_i'' \in \Theta_i(a_i^*)$, then any valuation that satisfies $\theta_i \in [\theta_i', \theta_i'']$ also satisfies $\theta_i \in \Theta_i(a_i^*)$.[6] This follows the definition of a convex subset of an ordered set (the set of valuations), which is slightly different from the more familiar use of "convex" for real numbers. In this use of "convex," θ_i is restricted to be from the set of valuations, rather than just an arbitrary number. For example, the set $\{1, 2, 3\}$ is a convex set of valuations if the valuations are restricted to be integers (using the natural ordering of the integers).

Similar to Assumptions 3 and 4, Assumption 5 accommodates the possibility that only some players use monotone strategies. However, for some parts of the sharpness result, and for an application of Lemma 1, it needs to be assumed that all players use monotone strategies, although they need

[6]Suppose that $a_i(\theta_i') = a_i^*$ and $a_i(\theta_i'') = a_i^*$. Suppose without loss of generality that $\theta_i' \leq \theta_i''$. Since $a_i(\cdot)$ is weakly increasing, any valuation between θ_i' and θ_i'' also uses action a_i^*.

not be "optimal" strategies. That would just require that players with larger valuations take larger actions, but not necessarily "optimal" actions.

3.3. Monotone effect of opponents' actions on utility. The identification strategy also uses another monotonicity assumption, closely related to Assumption 5 (Weakly increasing strategy is used). There are multiple versions of this assumption, and only one needs to be assumed. This is indicated by asterisks in the assumption numbering.

One version is the "primitive conditions" version, discussed in Section 3.3.1. Essentially, this version requires that *ex post* expected utility is a weakly decreasing function of the other players' actions, and that valuations are suitably positively dependent. Recall that positively dependent valuations (in particular, affiliated valuations) is commonly assumed in the economic theory literature that establishes sufficient conditions for Assumption 5.

Another version is the "high level conditions" version, discussed in Section 3.3.2. Essentially, this version requires that *ex interim* expected utility would be greater when a player "counterfactually" has the beliefs associated with valuation θ_i' as compared to the beliefs associated with valuation θ_i, when $\theta_i' \leq \theta_i$.

Both assumptions are discussed in more detail in the following sections. And, as discussed in Section 3.3.3, the "primitive conditions" (alongside the other assumptions of the identification strategy, in particular Assumptions 4 and 5) imply the "high level conditions." In principle, the "high level conditions" could be true even if the "primitive conditions" are false.

As formalized in the subsequent discussion, the "high level conditions" version is necessarily true in the special case of Assumption 1* (Independent valuations). Therefore, overall, this section is "irrelevant" under Assumption 1* (Independent valuations).

3.3.1. Primitive conditions. This section discusses the version of the assumption on model primitives.

Definition 1 (A subset of optimality-relevant actions)**.** For a number v_i that could be the valuation of player i, a set $\tilde{\mathcal{A}}_i(v_i) \subseteq \mathcal{A}_i$ is a *subset of optimality-relevant actions* if it satisfies

$$\sup_{z_i \in \mathcal{A}_i} \Big(v_i E_{Q_i}(\overline{x}_i(z_i, a_{-i})) - E_{Q_i}(\overline{t}_i(z_i, a_{-i})) \Big) = \sup_{z_i \in \tilde{\mathcal{A}}_i(v_i)} \Big(v_i E_{Q_i}(\overline{x}_i(z_i, a_{-i})) - E_{Q_i}(\overline{t}_i(z_i, a_{-i})) \Big)$$

for every distribution Q_i over $\mathcal{A}_{-i}$.

By construction, $\tilde{\mathcal{A}}_i(v_i)$ is a set of actions in which an optimal action for valuation v_i is guaranteed to exist, regardless of the distribution Q_i over $\mathcal{A}_{-i}$. The distribution Q_i can be interpreted as player i's beliefs over $\mathcal{A}_{-i}$; since v_i is fixed in Definition 1, implicitly this allows for the possibility of dependent valuations where players with different valuations have different beliefs. It is always true that $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i$ is a subset of optimality-relevant actions. In many games, a smaller set of actions can also be a subset of optimality-relevant actions. $\mathcal{A}_i \backslash \tilde{\mathcal{A}}_i(v_i)$ can be interpreted as a set of "weakly dominated" actions for the given valuation v_i. Later assumptions are specified only for a subset of optimality-relevant actions, so those assumptions are "weaker" when stated on a "smaller" subset of optimality-relevant actions. It will turn out this is critical for the suitability of the assumptions in particular games.

As the following examples illustrate, it is often trivial to find a subset of optimality-relevant actions that is a strict subset of $\mathcal{A}_i$. In particular, in many games, $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$ is a subset of optimality-relevant actions per Definition 1. That is because in many games it is "weakly dominated" for a player to take an action above its valuation. The examples exhibit a certain pattern in discussing the conditions and assumptions across the different specific games, which reflects that fundamentally the assumptions hold "in general" for a wide class of games, so the main question is providing concrete interpretations in concretely specified games.

Example 1 (Contests, continuing from p. 16)**.** If all players transfer their "effort," the *ex post* utility function becomes $v_i \overline{x}_i(z_i, a_{-i}) - z_i$. If only the winning player transfers its "effort," the *ex post* utility function becomes $(v_i - z_i)\overline{x}_i(z_i, a_{-i})$. Regardless of the distribution Q_i over $\mathcal{A}_{-i}$, player i with valuation v_i will always find it (weakly) optimal to use an effort weakly less than v_i: Consider an effort strictly greater than v_i. If the player wins, it "gets" v_i and transfers strictly more than v_i. If the bidder loses, it "gets" 0 and transfers at least 0 (depending on whether all players transfer their "effort"). Thus, with an effort strictly greater than v_i, the player gets at most 0 utility. This can also be accomplished by using effort of 0. Thus, the player will always find it (weakly) optimal to use effort weakly less than v_i. Thus, $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$ is a subset of optimality-relevant actions per Definition 1.

Example 2 (Auctions, continuing from p. 16)**.** Consider first-price auctions (allowing complications like reserve prices, participation costs, or multiple units). Regardless of the distribution Q_i over $\mathcal{A}_{-i}$, player i with valuation v_i will always find it (weakly) optimal to place a bid weakly less than v_i:

Consider a bid strictly greater than v_i. If the bidder wins, it "gets" v_i and transfers strictly more than v_i (which could involve a participation cost). If the bidder loses, it "gets" 0 and transfers at least 0 (which could involve a participation cost). Thus, with a bid strictly greater than v_i, the bidder gets at most 0 utility. This can also be accomplished by placing a bid of 0 (or not participating in the auction). Thus, the bidder will always find it (weakly) optimal to place a bid weakly less than v_i. Thus, $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$ is a subset of optimality-relevant actions per Definition 1. Note that the same basic logic applies to some other auction models, for example all-pay auctions. So in many auction formats, $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$.

Then, consider the following assumption.

Assumption 6** (Monotone *ex post* utility and positively dependent valuations)**.** Suppose either:

(a) For each $i \in \mathcal{J}$, for any number v_i that could be the valuation of player i (i.e., respecting non-negativity, when applicable) there is a subset of optimality-relevant actions $\tilde{\mathcal{A}}_i(v_i) \subseteq \mathcal{A}_i$ with the property that for any $z_i \in \tilde{\mathcal{A}}_i(v_i)$, it holds that $v_i \bar{x}_i(z_i, a_{-i}) - \bar{t}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i}.

(b) Assumption 6**(a) and Definition 1 holds with "for any distribution Q_i over $\mathcal{A}_{-i}$" replaced by "for any belief $\Pi_i(a_{-i}|\theta_i)$ over $\mathcal{A}_{-i}$ associated with some valuation θ_i" and "weakly decreasing function of a_{-i}" is replaced by "weakly decreasing function of a_{-i} restricted to a_{-i} from $\cup_{\theta_i} \text{support}\{\Pi_i(a_{-i}|\theta_i)\}$."

Suppose either:

(c) Valuations are affiliated.

(d) For each $i \in \mathcal{J}$, the distribution of $\theta_{-i}|(\theta_i = \theta_i')$ is stochastically smaller than the distribution of $\theta_{-i}|(\theta_i = \theta_i'')$ in the usual multivariate stochastic order, when $\theta_i' \leq \theta_i''$.

And suppose:

(e) The strategies satisfy the constraints $a_i(v_i) \in \tilde{\mathcal{A}}_i(v_i)$ for all valuations and players $i \in \mathcal{J}$.

Assumption 6**(a) requires that *ex post* expected utility is a weakly decreasing function of the actions of the other players, when player i with valuation v_i takes an action from $\tilde{\mathcal{A}}_i(v_i)$.[7] Thus, this condition requires that player i gets greater *ex post* utility when the other players take lower actions. This primitive condition can be easily checked in applications, as the following examples

[7]It is therefore somewhat similar to the assumption that the interaction effects are non-positive in models of market entry.

illustrate. Obviously, it would be more than sufficient that *ex post* expected utility is a weakly decreasing function of the actions of the other players regardless of the action taken by player i, setting $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i$. However, in certain games, that "weakly decreasing" condition holding for all z_i would be too strong. That is why the assumption only requires "weakly decreasing" when evaluated at $z_i \in \tilde{\mathcal{A}}_i(v_i)$. Assumption 6**(b) is a weaker condition, as is immediately clear from the statement.

Assumptions 6**(c) and 6**(d) are two standard "positive dependence" assumptions on the distribution of valuations. Whether or not these conditions hold is unrelated to the specific game being considered, as this condition concerns exclusively the valuations. Only one "positive dependence" condition is needed for the identification result; possibly, other "positive dependence" conditions would also suffice. For Assumption 6**(c), this paper uses the definition of "affiliated" from the auction theory literature (e.g., Milgrom and Weber (1982) or Milgrom (2004)). Assumption 6**(d) can be viewed as a multivariate version of positive regression dependence (e.g., Lehmann (1966, Section 5) or Shaked and Shanthikumar (2007, page 412)).

Assumption 6**(e) rules out the use of strategies that involve using the weakly dominated actions in $\mathcal{A}_i \backslash \tilde{\mathcal{A}}_i(v_i)$. As the following examples illustrate, although this is an assumption even given Assumption 3 (Optimal strategy is used), it is a mild (and standard) assumption.

Obviously, Assumption 6**(a) implies Assumption 6**(b). Nevertheless, both are given as sufficient conditions. Only Assumption 6**(b) is actually used in the analysis, but this depends on "true" beliefs. Assumption 6**(a) emphasizes the fact that the condition doesn't need to refer to any particular properties of "true" beliefs.

The following examples show the establishing of Assumption 6**(a). As discussed above, Assumptions 6**(c) and 6**(d) is just an assumption directly on valuations, so unrelated to the specifics of particular games. The following also illustrates Assumption 6**(e).

Example 1 (Contests, continuing from p. 19)**.** If all players transfer their "effort," *ex post* utility $v_i\overline{x}_i(z_i, a_{-i}) - z_i$ is a weakly decreasing function of a_{-i} given that $\overline{x}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i}. Note that in this case, no restriction on z_i related to the subset of optimality-relevant actions is needed. Alternatively, if only the winning players transfer its "effort," *ex post* utility $(v_i - z_i)\overline{x}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i} given that $\overline{x}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i}, *as long as* $z_i \in \tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$. This establishes Assumption 6**(a). This also illustrates the need for the $\tilde{\mathcal{A}}_i(v_i)$ construction. For $z_i > v_i$, actually $(v_i - z_i)\overline{x}_i(z_i, a_{-i})$ is a weakly *increasing* function of a_{-i}. In this case, the player has (irrationally) put forth more "effort" than its own valuation, so it "wants" to lose the contest (and be "refunded" its effort).

Given this subset of optimality-relevant actions, Assumption 6**(e) is the condition that players use strategies that do not put forth more effort than their own valuations. This is a (mild) assumption, even given Assumption 3 (Optimal strategy is used). Specifically in contests where losers are refunded their effort, a player could optimally put forth more effort than its valuation if it knows it will lose for sure given that effort (and thus be refunded its effort). In such a case, the player "should" just put forth 0 effort, per Assumption 6**(e).

Example 2 (Auctions, continuing from p. 19)**.** Again consider a first-price auction (allowing complications like reserve prices, participation costs, or multiple units). Given the condition that $z_i \in \tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$, Assumption 6**(a) requires that $v_i \overline{x}_i(z_i, a_{-i}) - \overline{t}_i(z_i, a_{-i}) = (v_i - z_i)\overline{x}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i}. This is true given that $\overline{x}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i} and given $z_i \leq v_i$. This establishes Assumption 6**(a). This again illustrates the need for the $\tilde{\mathcal{A}}_i(v_i)$ construction. For $z_i > v_i$, actually $(v_i - z_i)\overline{x}_i(z_i, a_{-i})$ is a weakly *increasing* function of a_{-i}. In this case, the player has (irrationally) bid more than its own valuation, so it "wants" to lose the auction.

Alternatively, for example in the case of an all-pay auction, the relevant condition becomes $v_i \overline{x}_i(z_i, a_{-i}) - z_i$ is a weakly decreasing function of a_{-i}. This is weakly decreasing in a_{-i}, in any standard auction where the high bid is allocated the object. This holds for all z_i.

Given this subset of optimality-relevant actions, Assumption 6**(e) is the condition that players use strategies that do not bid more than their own valuations. This is a (mild) assumption, even given Assumption 3 (Optimal strategy is used), because a player could optimally bid more than its valuation if it knows it will lose for sure given that bid. In such a case, the player "should" just bid 0 (or not participate in the auction), per Assumption 6**(e). Indeed, this is a common assumption in the auction theory literature, to rule out "unreasonable" bidding, e.g., Maskin and Riley (2003, Assumption 1) or Lebrun (2006, Theorem 1, etc.).

3.3.2. High level conditions. This section discusses the high level conditions version of the assumption. It involves the strategy $a_i(\theta_i)$ and the beliefs Π_i.

Assumption 6 (Monotone effect of counterfactual beliefs on utility)**.** *It holds that:*

(a) *For each $i \in \mathcal{J}$, and any possible valuations $\theta_i' \leq \theta_i$, $a_i(\theta_i)$ from Assumption 5(a) satisfies*

$$\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i), a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i), a_{-i})|\theta_i') \geq \theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i), a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i), a_{-i})|\theta_i).$$

(b) *For each $i \in \mathcal{J}$, and any possible valuations $\theta_i \leq \theta_i''$,*

$$\sup_{z_i \in \mathcal{A}_i} (\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)) \geq \sup_{z_i \in \mathcal{A}_i} (\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i'')).$$

Assumption 6 is satisfied if valuations are independent, since then beliefs do not depend on the valuation. By construction, $E_{\Pi_i}(\overline{x}_i(a_i(\theta_i), a_{-i})|\theta_i') = \int \overline{x}_i(a_i(\theta_i), a_{-i}) d\Pi_i(a_{-i}|\theta_i')$, with similar expressions for the other terms in Assumption 6.

A simpler but stronger[8] assumption is the following:

Assumption 6* (Monotone effect of counterfactual beliefs on utility, simpler version)**.** For each $i \in \mathcal{J}$, and any possible valuations $\theta_i' \leq \theta_i$, there is a subset of optimality-relevant actions $\tilde{\mathcal{A}}_i(\theta_i) \subseteq \mathcal{A}_i$ such that

$$\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i') \geq \theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)$$

for all $z_i \in \tilde{\mathcal{A}}_i(\theta_i)$. And Assumption 6**(e) holds.

Assumption 6* has a simple interpretation: the payoff that a player gets from a given action $z_i \in \tilde{\mathcal{A}}_i(\theta_i)$ is greater when it has the beliefs associated with valuation θ_i' as compared to when it has the beliefs associated with valuation θ_i with $\theta_i \geq \theta_i'$. The beliefs determine the "strength" of the opponents' actions. Given Assumption 5 (Weakly increasing strategy is used), a player with the beliefs of valuation θ_i with $\theta_i \geq \theta_i'$ believes that its opponents will stochastically take "stronger" actions (e.g., higher bids in auctions, higher efforts in contests, etc.) as compared to when it has the beliefs associated with valuation θ_i'. In other words, Assumption 6* states that players are "better off" when they face a "weaker" distribution of actions. Recall from the above discussion that $\tilde{\mathcal{A}}_i(\theta_i) = \mathcal{A}_i$ is a subset of optimality-relevant actions. The allowed restriction to $z_i \in \tilde{\mathcal{A}}_i(\theta_i)$ is important because the inequality in Assumption 6* may not hold in some games at some "irrational" actions z_i such that, if a player takes that action z_i, it actually prefers the opponents to take "stronger" actions.

[8]It is obvious that Assumption 6* implies Assumption 6(a). Assumption 6(b) also is implied because $\sup_{z_i \in \mathcal{A}_i}(\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)) \geq \theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i) \geq \theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i'')$ for all $z_i \in \tilde{\mathcal{A}}_i(\theta_i)$.

Assumption 6 is weaker still, but shares the same basic structure. Assumption 6(a) requires the inequality from Assumption 6 hold only specifically at $z_i = a_i(\theta_i)$. Assumption 6(b) requires the inequality from Assumption 6 hold only specifically for the "optimal" possible utility.

Example 1 (Contests, continuing from p. 21)**.** Consider a contest from the point of view of a player with valuation θ_i. Given that $a_i(\theta_i) \leq \theta_i$, the player "wants" to win the contest. This is more likely when the opponents put forth less effort, which under the use of monotone strategies, arises under the beliefs associated with valuation θ_i' with $\theta_i' \leq \theta_i$ as compared to the beliefs associated with valuations θ_i. This implies Assumption 6(a). By similar reasoning, the maximal possible utility that the player can achieve with the beliefs associated with valuation θ_i with $\theta_i \leq \theta_i''$ is weakly greater than the maximal possible utility that the player can achieve with the beliefs associated with valuation θ_i'', since the former concerns "weaker" opponents. This establishes Assumption 6(b). More formally, Assumption 6 can be established using Lemma 1 in light of the previous discussion of contests and Assumption 6**.

Example 2 (Auctions, continuing from p. 22)**.** Consider a first-price auction from the point of view of a player with valuation θ_i. Given that $a_i(\theta_i) \leq \theta_i$, the player "wants" to win the auction. This is more likely when the opponents place lower bids, which under the use of monotone strategies, arises under the beliefs associated with valuation θ_i' with $\theta_i' \leq \theta_i$ as compared to the beliefs associated with valuations θ_i. This implies Assumption 6(a). By similar reasoning, the maximal possible utility that the player can achieve with the beliefs associated with valuation θ_i with $\theta_i \leq \theta_i''$ is weakly greater than the maximal possible utility that the player can achieve with the beliefs associated with valuation θ_i'', since the former concerns "weaker" opponents. This establishes Assumption 6(b). More formally, Assumption 6 can be established using Lemma 1 in light of the previous discussion of auctions and Assumption 6**.

Remark 2 (Interpretation of the terms in Assumption 6)**.** The left side of the inequality in Assumption 6(a) is the *ex interim* expected utility experienced by player i that has valuation θ_i that uses action $a_i(\theta_i)$ and "counterfactually" has the beliefs of valuation $\theta_i' \leq \theta_i$. Changing only the beliefs part of this expression, the right side of the inequality in Assumption 6(a) is the *ex interim* expected utility experienced by player i that has valuation θ_i that uses action $a_i(\theta_i)$ and has the beliefs of valuation θ_i. Assumption 6(b) is similar. The left side of Assumption 6(b) is the "optimal" *ex interim* expected utility experienced by player i that has valuation θ_i and has the beliefs of valuation θ_i. The right side of the inequality in Assumption 6(b) is the supremum of the possible *ex interim* expected utilities experienced by player i that has valuation θ_i that uses some action $z_i \in \mathcal{A}_i$ and "counterfactually" has the beliefs of valuation $\theta_i'' \geq \theta_i$. Therefore, an interpretation of Assumption 6 is that "counterfactual" *ex interim* expected utility is suitably weakly decreasing in the valuation that generates the "counterfactual" beliefs.

3.3.3. Relationship between conditions. Assumption 6** (alongside the other assumptions already used in the analysis) implies Assumption 6.

Lemma 1 (Assumption 6** implies Assumption 6)**.** *Suppose Assumption 6** (Monotone* ex post *utility and positively dependent valuations). Suppose Assumption 4 (Correct beliefs) is satisfied. Suppose each player $i \in \mathcal{I}$ uses a weakly increasing pure strategy, which need not be optimal; for this, it is more than sufficient that Assumption 5 (Weakly increasing strategy is used) is satisfied with $N_1 = N$. Then Assumption 6 is satisfied.*

Besides Assumption 6**, the conditions in Lemma 1 basically reiterate previous assumptions. Assumption 4 is assumed. It is assumed that *all* players use a weakly increasing strategy, though this strategy does not need to be optimal for any players $i \notin \mathcal{J}$. Those players must take actions that are a weakly increasing function of their valuation, but this need not be an "optimal" action.

Remark 3 (Opposite direction of monotonicity)**.** If the direction of the monotonicity happens to be opposite that of Assumption 6, it is straightforward to adjust the identification result accordingly (essentially the inequality $z_i' < a_i < z_i''$ switches directions in the statement of Theorem 1).

Remark 4 (Intuition for Lemma 1)**.** These sufficient conditions combine to justify the following argument, which suffices for Assumption 6. Consider holding fixed the valuation that a player "actually experiences" for the object. This is as in the statement of Assumption 6, where θ_i is held fixed in all parts of the statement of the assumption. Then, as in the statement of Assumption 6(a), consider the impact on expected utility of having the beliefs $\Pi_i(\cdot|\theta_i')$ compared to $\Pi_i(\cdot|\theta_i)$, where $\theta_i' \leq \theta_i$. By Assumptions 6**(c) and 6**(d), player i believes that the other players tend to have higher valuations under $\Pi_i(\cdot|\theta_i)$ as compared to under $\Pi_i(\cdot|\theta_i')$. Given the use of monotone strategies, this implies that player i believes that the other players tend to use higher actions under $\Pi_i(\cdot|\theta_i)$ as compared to under $\Pi_i(\cdot|\theta_i')$. By the assumption that *ex post* expected utility is decreasing in the other players' actions, this means that player i is worse off under $\Pi_i(\cdot|\theta_i)$ as compared to under $\Pi_i(\cdot|\theta_i')$, exactly in the sense required by Assumption 6(a). It is similar for Assumption 6(b).

3.4. Ex ante bounds. As often with partial identification results, the identification result can account for an *ex ante* known lower bound or upper bound for the partially identified quantity.

Assumption 7 (Known bounds on valuations)**.** *For each $i \in \mathcal{J}$, θ_i must be in the set $[\Theta_{Li}, \Theta_{Ui}]$.*

Assumption 7 is the statement that the support of the valuations is *contained within* $[\Theta_{Li}, \Theta_{Ui}]$.[9] The econometrician need not know the support of the valuations. Assumption 7 allows the econometrician to impose knowledge that θ_i is at least Θ_{Li} and no more than Θ_{Ui}, even before observing the data. In many games, it might be reasonable to set $\Theta_{Li} = 0$, reflecting that the object is known to have non-negative value to all players. By setting $\Theta_{Li} = -\infty$ and $\Theta_{Ui} = \infty$, it is possible to check the identification result without such known bounds.

Assumption 8 (Known bounds on actions)**.** *It holds that:*

(a) *For each $i \in \mathcal{J}$, for any number v_i that could be the valuation of player i, regardless of the beliefs Q_i over $\mathcal{A}_{-i}$ held by player i, player i with valuation v_i uses an action from the set $\mathcal{A}_i \cap [a_{Li}(v_i), a_{Ui}(v_i)]$.*

(b) *For each $i \in \mathcal{J}$, $a_{Li}(\cdot)$ is either $-\infty$ or a known continuous*[10] *weakly increasing real-valued function and $a_{Ui}(\cdot)$ is either ∞ or a known continuous weakly increasing real-valued function.*

Assumption 8 allows the econometrician to impose knowledge of basic properties of the actions used by the players. For example, in first-price auctions, a plausible specification is $a_{Li}(v_i) \equiv \inf \mathcal{A}_i$ and $a_{Ui}(v_i) \equiv v_i$, based on the previous discussion of the fact that a player's optimal bid can always be taken to be weakly less than the player's valuation. The same would be true in many other games. Similar to Assumption 7, by setting $a_{Li}(v_i) \equiv -\infty$ and $a_{Ui}(v_i) \equiv \infty$, it is possible to check the identification result without such known properties. (In fact, this would be equivalent to setting $a_{Li}(v_i) \equiv \inf \mathcal{A}_i$ and $a_{Ui}(v_i) \equiv \sup \mathcal{A}_i$.) Define $a_{Li}^{-1}(a_i) = \sup\{v_i : a_{Li}(v_i) \leq a_i\}$ and $a_{Ui}^{-1}(a_i) = \inf\{v_i : a_{Ui}(v_i) \geq a_i\}$.

[9] A similar sort of assumption is commonly used in the partial identification of treatment effects, where it is commonly assumed that the responses must be within a known range, while not requiring that all responses within that range are actually achieved.

[10] Continuity is only used in the sharpness results. In fact, the sharpness results only use the condition that $a_{Li}(a_{Li}^{-1}(z_i)) \leq z_i$ and $a_{Ui}(a_{Ui}^{-1}(z_i)) \geq z_i$, for all z_i, which is implied by continuity.

For example, when $a_{Li}(v_i) \equiv \inf \mathcal{A}_i$ and $a_{Ui}(v_i) \equiv v_i$, $a_{Li}^{-1}(a_i) = \sup\{v_i : \inf \mathcal{A}_i \leq a_i\} = \sup \mathbb{R} = \infty$ and $a_{Ui}^{-1}(a_i) = \inf\{v_i : v_i \geq a_i\} = a_i$. The specification $a_{Li}(v_i) \equiv \inf \mathcal{A}_i$ and $a_{Ui}(v_i) \equiv v_i$ has no identifying power for the *upper* bound for the distribution of valuations.

4. Identification results

4.1. Definition of stochastic ordering. The identification strategy results in bounds on the multivariate distribution of valuations in terms of the usual multivariate stochastic order.

Definition 2 (Upper set)**.** Let $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$. A set U is an upper set if $x \in U$ and $y \geq x$ implies that $y \in U$.

Definition 3 (Usual multivariate stochastic order)**.** Let A and B be d-dimensional random vectors, with probability laws P_A and P_B. A is stochastically larger than B in the usual multivariate stochastic order if $P_A(U) \geq P_B(U)$ for all Borel measurable upper sets U.

As formalized in Shaked and Shanthikumar (2007, Theorem 6.B.1), and more generally in Kamae et al. (1977) for partially ordered Polish spaces, random vector A is stochastically larger than random vector B in the usual multivariate stochastic order exactly when there are $\hat{A}$ and $\hat{B}$ defined on the same probability space, such that $\hat{A}$ has the same distribution as A and $\hat{B}$ has the same distribution as B, and such that $\hat{A} \geq \hat{B}$ with probability 1. Indeed, it is obvious that this condition on $\hat{A}$ and $\hat{B}$ implies Definition 3, which is the logic sequence used in the identification strategy.

The partial identification result establishes that the random vector of valuations $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is stochastically larger than a certain random vector and is stochastically smaller than another certain random vector. The random vectors that are the upper and lower bounds for θ are themselves identified quantities, and have a constructive definition as a function of the observable data.

As discussed in Shaked and Shanthikumar (2007, Chapter 6), and more generally in Kamae et al. (1977), the condition that random vector A is stochastically larger than random vector B in the usual multivariate stochastic order is equivalent to the condition that $E(\phi(A)) \geq E(\phi(B))$ for all weakly increasing functions ϕ for which the expectations exist. In particular, because $\phi(X) = 1[X \leq t]$ is weakly decreasing in X, the condition that A with distribution function F_A is stochastically larger than B with distribution function F_B in the usual multivariate stochastic order implies that $F_A(t) \leq F_B(t)$ for all t. The condition that $F_A(t) \leq F_B(t)$ for all t is known as the lower orthant order (e.g., Shaked and Shanthikumar (2007, Chapter 6.G.1)). The lower orthant order is a distinct sense of stochastic ordering. For random vectors, unlike for scalar random variables, the lower orthant ordering is implied by, but does not imply, the usual multivariate stochastic ordering.

Bounds on the distribution of valuations in the usual multivariate stochastic order also imply bounds on other quantities derived from the distribution of valuations, as discussed in Shaked and Shanthikumar (2007, Chapter 6). In particular, ordinary random vector A stochastically larger than ordinary random vector B in the usual multivariate stochastic order implies that a given order statistic from A is stochastically larger than the same order statistic from B, by applying Shaked and Shanthikumar (2007, Theorem 6.B.16 or Theorem 6.B.23). In their independent private values English auction setup, Haile and Tamer (2003) have shown how to use lower orthant bounds on the scalar distribution of valuations to bound the optimal reserve price.

4.2. Game-structure identification of differences. An important step in the identification strategy concerns identifying the rules of the game. If the econometrician has *ex ante* knowledge of the rules, there is basically nothing to do on this point. This section allows the case that the econometrician may not have *ex ante* knowledge of the rules. It is possible to point identify the rules from the observed data, under weak conditions.

Let $\mathcal{A}_i^d$ be the support of A_i, the actions used in the data. $\mathcal{A}_i^d$ might be a proper subset of $\mathcal{A}_i$.

Definition 4 (Game-structure identification of differences)**.** The specification $(a_i, z_i, z_i', z_i'') \in \mathcal{A}_i^4$ with $z_i', z_i'' \in \mathcal{A}_i^d$ is a specification with game-structure identification of differences if

$$E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') \text{ and } E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$$

are point identified. The set of specifications with game-structure identification of differences is $\mathcal{R}_i$.

By construction, $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') = \int \overline{x}_i(a_i, A_{-i})dP(A_{-i}|A_i = z_i')$, with similar expressions for the other terms in Definition 4. That is, this concerns the integral of $\overline{x}_i(a_i, A_{-i})$ viewed as a function of A_{-i}, with respect to the observed distribution $A_{-i}|(A_i = z_i')$.

Each specification in $\mathcal{R}_i$ is a specification for which it is possible to evaluate the difference, for any given valuation θ_i, between two specific payoffs: the payoff from the action a_i given that the players $-i$ use the distribution of actions $A_{-i}|(A_i = z_i')$ and the payoff from the action z_i given that the players $-i$ use the distribution of actions $A_{-i}|(A_i = z_i'')$. The reason this particular comparison is relevant will become more clear in the sketch of the identification strategy in Section 4.4. A general feature of the identification strategy is that each additional specification in $\mathcal{R}_i$ provides additional "identified restrictions" on the valuation that is consistent with a given observed action. Having relatively more specifications in $\mathcal{R}_i$ is a *necessary* but not *sufficient* condition for the identified bounds to be more informative. Even if $\mathcal{R}_i$ is as large as possible (i.e., $\mathcal{R}_i = \mathcal{A}_i^4$), it is possible there is not point identification of valuations, for example due to "flat spots" in the strategies.

This definition allows both for the possibility that the econometrician has *ex ante* knowledge of the rules, and the possibility that the econometrician recovers them from the observed data.

One sufficient condition for game-structure identification of differences at any given specification (a_i, z_i, z_i', z_i'') is for the allocation rule and transfer rule to be known *ex ante* (before observing the data) by the econometrician. This is true because Definition 4 involves expected values of the allocation rule and transfer rule with respect to the observed distribution of $P(A)$. Thus, game-structure identification of differences can fail at a particular specification only when the econometrician does not *ex ante* know the allocation rule and/or transfer rule.

In fact, for particular functional forms of the allocation rule and transfer rule, it suffices for the econometrician to know less than the entire rule, since only differences are relevant for Definition 4. For instance, this can accommodate an unknown (to the econometrician) participation cost.[11]

The econometrician knowing the rules of the game is the standard setup in identification in structural econometrics. The rest of this section explains that it is possible to use the data to learn the rules of the game. One additional requirement of that approach is that the observed data must include the realized allocations and realized transfers, rather than just the realized actions.

This is because $\overline{x}_i(a_i, a_{-i}) = E_P(X_i | A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a_i, a_{-i}) = E_P(T_i | A_i = a_i, A_{-i} = a_{-i})$ are point identified quantities under standard conditions on identification/estimation of conditional expectations. The following lemma straightforwardly formalizes such standard conditions for game-structure identification of differences. Let $\mathcal{A}^d$ be the support of the observed actions $(A_1, A_2, \ldots, A_N)$.

[11]In an auction, for example, a participation cost is a transfer paid by any bidder who places a bid (rather than taking the "do not participate" action). Suppose that $\overline{t}_i(a_i, a_{-i}) \equiv \overline{t}_{i1}(a_i, a_{-i}) + \overline{t}_{i2}(a_i, a_{-i}) \equiv \overline{t}_{i1}(a_i, a_{-i}) + \overline{t}_{i2}(a_i)$, so that the transfer is the sum of two transfers, one of which depends only on a_i. Then the relevant difference is $E_P(\overline{t}_i(a_i, A_{-i}) | A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i}) | A_i = z_i'') = \big(E_P(\overline{t}_{i1}(a_i, A_{-i}) | A_i = z_i') + \overline{t}_{i2}(a_i)\big) - \big(E_P(\overline{t}_{i1}(z_i, A_{-i}) | A_i = z_i'') + \overline{t}_{i2}(z_i)\big)$. It would therefore suffice for the econometrician to know $\overline{t}_{i1}(a_i, a_{-i})$ for all (a_i, a_{-i}) and $\overline{t}_{i2}(a_i) - \overline{t}_{i2}(z_i)$ at least for the specified (a_i, z_i). If $\overline{t}_{i2}$ is the participation cost, and the cost of participating is the same for participating actions a_i and z_i, then the econometrician knows that $\overline{t}_{i2}(a_i) - \overline{t}_{i2}(z_i) = 0$ even if the econometrician does not know the participation cost.

Lemma 2 (Sufficient conditions for game-structure identification of differences with unknown allocation rule and/or transfer rule)**.** *Suppose that Assumptions 1 (Dependent valuations) and 2 (Action space is ordered) are satisfied. Suppose the data is $P(A, X, T)$. Suppose $E_P(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i, A_{-i} = a_{-i})$ are point identified for any $a \in \mathcal{A}^d$. Suppose the distribution $A_{-i}|(A_i = a_i)$ is point identified for any $a_i \in \mathcal{A}_i^d$. And suppose $\mathcal{A}^d = \prod_i \mathcal{A}_i^d$. Then $\overline{x}_i(a_i, a_{-i})$ and $\overline{t}_i(a_i, a_{-i})$ are point identified for any $a_i \in \mathcal{A}_i^d$ and $a_{-i} \in \mathcal{A}_{-i}^d$. And, then, $E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i')$ and $E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i')$ are point identified for any $z_i \in \mathcal{A}_i^d$ and $z_i' \in \mathcal{A}_i^d$. And, then, any specification of actions $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i^d)^4$ is a specification with game-structure identification of differences per Definition 4.*

This result requires point identification of certain observable conditional expectations and conditional distributions. Relative to the identification literature, it is standard to use this as a primitive condition on the population data.[12]

If the econometrician uses the data to point identify the allocation rule and transfer rule, rather than knows them *ex ante*, then only specifications (a_i, z_i, z_i', z_i'') with $a_i \in \mathcal{A}_i^d$ and $z_i \in \mathcal{A}_i^d$ will have game-structure identification of differences per Lemma 2. Fewer specifications in $\mathcal{R}_i$ results in relatively wider identified bounds.

Independent valuations. *Under Assumption 1* (Independent valuations), A_{-i} is independent of A_i and therefore z_i' and z_i'' effectively play no role in Definition 4. So, under Assumption 1*, a specification $(a_i, z_i) \in \mathcal{A}_i^2$ is a specification with game-structure identification of differences if it satisfies the condition in Definition 4, without the conditioning on z_i' and z_i''. Hence, under Assumption 1*, the dimension of elements of $\mathcal{R}_i$ changes. By notation, the set of specifications with game-structure identification of differences under Assumption 1* is $\mathcal{R}_i^{\perp}$.* ★

4.3. Identification results. To state the identification result, define the following terms. The reason these terms are relevant will be shown in the sketch of the identification strategy in Section 4.4. The expression in Equation 4 for $\Phi_{Li}(\cdot)$ involves $a_{Ui}^{-1}(\cdot)$; the differing subscripts are correct. Similar issues arise in other expressions.

[12]Under almost no assumptions, kernel regression estimators of conditional expectations are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable (e.g., Stone (1977), Devroye (1981), or Greblicki et al. (1984)). Under mild continuity assumptions, the result can be strengthened to show consistency for *all* realizations of the conditioning variable, as in Bierens (1987). And, kernel estimators of conditional distributions are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable, and all realizations of the conditioning variable if the conditional distribution depends in a suitably continuous way on the conditioning variable (e.g., Stute (1986), Owen (1987), and Hall et al. (1999)).

$$\Phi_{Li}^{(1)}(a_i) = \sup_{z_i, z_i', z_i''} \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} : \\ z_i' < a_i < z_i'', \\ z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\}, \\ (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \tag{3}$$

$$\Phi_{Li}(a_i) = \max\{\Phi_{Li}^{(1)}(a_i), \Theta_{Li}, a_{Ui}^{-1}(a_i)\} \tag{4}$$

$$\Phi_{Ui}^{(1)}(a_i) = \inf_{z_i, z_i', z_i''} \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} : \\ z_i' < a_i < z_i'', \\ z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}, \\ (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \tag{5}$$

$$\Phi_{Ui}(a_i) = \min\{\Phi_{Ui}^{(1)}(a_i), \Theta_{Ui}, a_{Li}^{-1}(a_i)\} \tag{6}$$

Note that if $\Theta_{Li} = -\infty$, then Θ_{Li} has no impact on $\Phi_{Li}(a_i)$, consistent with the previous discussion of Assumption 7. The same is true when $\Theta_{Ui} = \infty$. And a similar statement is true for Assumption 8.

Let

$$\Upsilon_{Li}(a_i) = \sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i') \text{ and } \Upsilon_{Ui}(a_i) = \inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i'). \tag{7}$$

Section 4.4 contains a sketch of the identification strategy, explaining the following main result.

Theorem 1. *Under Assumptions 1 (Dependent valuations), 2 (Action space is ordered), 3 (Optimal strategy is used), 4 (Correct beliefs), 5 (Weakly increasing strategy is used), 6 (Monotone effect of counterfactual beliefs on utility) or 6** (Monotone* ex post *utility and positively dependent valuations), 7 (Known bounds on valuations), and 8 (Known bounds on actions), the distribution of valuations $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ is partially identified, and the identification is constructive, because the distribution of $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ is stochastically larger than the distribution of* $(\Upsilon_{L1}(A_1), \Upsilon_{L2}(A_2), \ldots, \Upsilon_{LN_1}(A_{N_1}))$

and is stochastically smaller than the distribution of $(\Upsilon_{U1}(A_1), \Upsilon_{U2}(A_2), \ldots, \Upsilon_{UN_1}(A_{N_1}))$ *, in the sense of the usual multivariate stochastic order, where* $(A_1, A_2, \ldots, A_{N_1})$ *is distributed according to the data* $P(A, X, T)$ *and* $\Upsilon_{Li}(\cdot)$ *and* $\Upsilon_{Ui}(\cdot)$ *are the identifiable functions given in Equation 7.*

Independent valuations. *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 1* (Independent valuations), drop Assumption 6 (Monotone effect of counterfactual beliefs on utility) and Assumption 6** (Monotone* ex post *utility and positively dependent valuations), and replace the* Υ *functions with the* Γ *functions defined in Equation 12, below.*

(8)
$$\Xi^{(1)}_{Li}(a_i) = \sup_{z_i} \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})) - E_P(\bar{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))} : \\ \quad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) > 0\}, \\ \quad (a_i, z_i) \in \mathcal{R}_i^{\perp} \end{cases}$$

(9)
$$\Xi_{Li}(a_i) = \max\{\Xi^{(1)}_{Li}(a_i), \Theta_{Li}, a^{-1}_{Ui}(a_i)\}$$

(10)
$$\Xi^{(1)}_{Ui}(a_i) = \inf_{z_i} \begin{cases} \frac{E_P(\bar{t}_i(a_i, A_{-i})) - E_P(\bar{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))} : \\ \quad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) < 0\}, \\ \quad (a_i, z_i) \in \mathcal{R}_i^{\perp} \end{cases}$$

(11)
$$\Xi_{Ui}(a_i) = \min\{\Xi^{(1)}_{Ui}(a_i), \Theta_{Ui}, a^{-1}_{Li}(a_i)\}$$

(12)
$$\Gamma_{Li}(a_i) = \sup_{a'_i \leq a_i, a'_i \in \mathcal{A}^d_i} \Xi_{Li}(a'_i) \text{ and } \Gamma_{Ui}(a_i) = \inf_{a'_i \geq a_i, a'_i \in \mathcal{A}^d_i} \Xi_{Ui}(a'_i).$$

Further, under Assumption 1 (Independent valuations), game-structure identification of differences can be established:* $E_P(\overline{x}_i(z_i, A_{-i})) = E_P(X_i | A_i = z_i)$ *and* $E_P(\bar{t}_i(z_i, A_{-i})) = E_P(T_i | A_i = z_i)$*.* ★

Remark 5 (Relaxing equilibrium assumptions)**.** Bayesian Nash equilibrium requires that all players act rationally given beliefs (Assumption 3 with $N_1 = N$) and have correct beliefs (Assumption 4 with

$N_1 = N$). This assumption of equilibrium is standard, but in some settings it may be too strong.[13] In auction models, for example, it might be that some "novice" bidders do not satisfy those assumptions whereas "experienced" bidders do satisfy those assumptions. The difference between "novice" and "experienced" might be due to learning from participating in previous auctions, or some other reason that is observable by the econometrician, so that the econometrician can distinguish between "novices" and "experienced" players. For example, Hortaçsu and Puller (2008) find that "large" firms are more strategically sophisticated than "small" firms. When $N_1 < N$, the identification analysis assumes only that players 1 through N_1 have correct beliefs and act rationally given those beliefs. For example, those players could have correct beliefs that the other players are "irrational."

The identification strategy in this paper can also be extended to allow for more substantial violations of Assumption 3 (Optimal strategy is used).

Assumption 3* (ε-optimal strategy is used)**.** For each $i \in \mathcal{J}$, there is a known $\varepsilon_i \geq 0$, such that for each possible valuation θ_i, player i uses a pure strategy $a_i(\theta_i)$ when it has valuation θ_i, with $\theta_i E_{\Pi_i}(\overline{x}_i(a_i(\theta_i), a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(a_i(\theta_i), a_{-i})|\theta_i) + \varepsilon_i \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)\right)$, so the action taken according to the strategy $a_i(\theta_i)$ comes within ε_i utils of maximizing *ex interim* expected utility.

This assumption follows the idea of an ϵ-equilibrium from Radner (1980) and others. As illustrated in the sketch of the identification strategy in Section 4.4, it is easy to adjust the partial identification strategy to use Assumption 3* (ε-optimal strategy is used) rather than Assumption 3 (Optimal strategy is used). The resulting $\Phi^{(1)}$ and $\Xi^{(1)}$ in Theorem 1 are adjusted to have an additional $-\varepsilon_i$ in the numerator. (Because of the signs of the denominator, this results in a reduced lower bound and an increased upper bound.) Furthermore, ε_i would be added to the upper bound on "foregone" utility in Theorem 4. This identification result would still involve the assumption of the use of monotone strategies, that are "approximately" optimal per Assumption 3*.

Remark 6 (The effect of Assumption 1* (Independent valuations))**.** The identified bounds are basically not tightened by the use of Assumption 1* in the identification analysis, assuming that Assumption 1* holds in the data generating process. Rather, the functional form of the bounds are

[13] Identification relaxing the assumption of equilibrium, or related questions, has been considered in Aradillas-López and Tamer (2008), Haile et al. (2008), Kline and Tamer (2012), Kline (2015, 2018), Syrgkanis et al. (2018), Aguirregabiria and Magesan (2020), and Magnolfi and Roncoroni (2023). Kline (2018) includes a discussion of the tradeoffs between equilibrium assumptions and assumptions on the data, for identification in entry games. See Maskin (2011) for a commentary on Nash equilibrium.

simplified when imposing Assumption 1*. And, the identification result can be stated under a slightly reduced set of other assumptions. Assumption 6 and Assumption 6** can be dropped, because they are implied by Assumption 1*, as discussed in Section 3.3.

Note that this claim holds fixed the data generating process and varies the assumptions *imposed* by the econometrician. It is *not* about differences in the bounds depending on whether Assumption 1* (Independent valuations) is *satisfied* in the data generating process.

If the data generating process satisfies Assumption 1* (Independent valuations), actions are also independent, so the conditioning on A_i is dropped from the expressions for Φ in Equations 3 and 5. This would be true whether or not the econometrician imposes Assumption 1* on the identification analysis. In that case, Φ becomes almost the same as Ξ in Equations 8 and 10. One possible difference concerns the bounds at the smallest/largest actions used in the data, where Φ_L is the maximum of the *ex ante* lower bound and $a_{Ui}^{-1}(\cdot)$ evaluated at the action, and Φ_U is the minimum of the *ex ante* upper bound and $a_{Li}^{-1}(\cdot)$ evaluated at the action, due to the role of z_i' and z_i''. But Ξ may be a different value. Another possible difference concerns the possibly differential impact of the sets $\mathcal{R}_i$ and $\mathcal{R}_i^{\perp}$.[14] Since the rules can be identified from the data per Lemma 2, the only difference could be from *ex ante* knowledge. Combined with the ideas of Remark 8, this is unlikely to have much effect.

Remark 7 (Weakening the bounds)**.** The identified bounds involve dividing by $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$, which may be close to zero. Trivially, the bounds remain valid if there is a further restriction to values of (z_i, z_i', z_i'') such that this term is a pre-selected tolerance away from zero. This can be relevant for an empirical application of the bounds, similar to the trimming of derivatives in the derivative-based approach to identification in auctions (e.g., Guerre et al. (2000, page 541) or Li et al. (2002, page 180)).

Remark 8 (The effect of *ex ante* knowledge of the rules)**.** This remark provides an argument for why *ex ante* knowledge of the rules should be generally expected to have relatively modest tightening effect on the bounds, as compared to identification of the rules from the data. This is seen in the numerical illustration in Section 5.

[14]In principle, the set of $(a_i, z_i) \in \mathcal{R}_i^{\perp}$ could be different from the set of (a_i, z_i) consistent with a specification of $(a_i, z_i, z_i', z_i'') \in \mathcal{R}_i$, in which case the infs/sups in Φ would be over a different set of values of (a_i, z_i) compared to in Ξ. If $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$ and $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$ are point identified for given specification of (a_i, z_i, z_i', z_i''), then under the independence assumption also $E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))$ and $E_P(\overline{t}_i(a_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))$ are point identified. Thus, the set $\mathcal{R}_i^{\perp}$ contains every value of (a_i, z_i) consistent with an element of the set $\mathcal{R}_i$.

Per Lemma 2, the data can be used to ensure that $(\mathcal{A}_i^d)^4 \subseteq \mathcal{R}_i$. Moreover, by Definition 4, any element of $\mathcal{R}_i$ of the form (a_i, z_i, z_i', z_i'') is such that $\{z_i', z_i''\} \in \mathcal{A}_i^d$. And given the usage in the identification result, only specifications with $a_i \in \mathcal{A}_i^d$ are relevant. Therefore, *ex ante* knowledge of the rules *could* tighten the identified bounds only when it implies that a specification $z_i \notin \mathcal{A}_i^d$ is part of a specification of an element of $\mathcal{R}_i$.

This can be expected to be a modest (or zero) effect. The fact that $z_i \notin \mathcal{A}_i^d$ means that every valuation actually in the real data has an associated utility maximizing action that is not z_i. Otherwise, z_i would be used in the data. In a very general sense, the identification strategy recovers bounds on the distribution of valuations from the utility maximization problem facing each player. As such, ignoring these "irrelevant" actions can be expected to have modest effect.

However, there are some caveats to this. Actually, the identification strategy must take a somewhat indirect approach to using the utility maximization problem facing each player, in particular because beliefs are unknown to the econometrician. It is possible that this "indirect approach" ends up with tighter bounds when the rules are *ex ante* known even for "irrelevant" z_i, because such z_i may not actually be "irrelevant" relative to the "indirect approach" to the utility maximization problem. Also, a valuation that is possible according to a given distribution within the identified set may not actually exist in the real data. (That is, the identification result does not generally point identify the support of valuations.) In that case, that valuation might actually not find such an action z_i to be "irrelevant" even though all the "real" valuations did find it to be irrelevant.

Remark 9 (Possibility of uninformative bounds)**.** It is possible that $\Phi_{Li}(a_i)$ is uninformative at some a_i, or that $\Phi_{Ui}(a_i)$ is uninformative at some a_i. Specifically, $\Phi_{Li}(a_i) = -\infty$ exactly when $\Theta_{Li} = -\infty$, $a_{Ui}^{-1}(a_i) = -\infty$, and there is no $(a_i, z_i, z_i', z_i'') \in \mathcal{R}_i$ with $z_i' < a_i < z_i''$ and $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0$. And, $\Phi_{Ui}(a_i) = \infty$ exactly when $\Theta_{Ui} = \infty$, $a_{Li}^{-1}(a_i) = \infty$, and there is no $(a_i, z_i, z_i', z_i'') \in \mathcal{R}_i$ with $z_i' < a_i < z_i''$ and $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0$. Similarly, it is possible that the actual lower bound $\Gamma_{Li}(a_i)$ is uninformative at some a_i, or that the actual upper bound $\Gamma_{Ui}(a_i)$ is uninformative at some a_i, when the above holds for all weakly smaller actions used in the data, or weakly larger actions used in the data, respectively, per Equation 12. If this happens, then the lower bound distribution and/or the upper bound distribution is actually an extended real valued random variable, taking on the values $\pm\infty$ with positive probability. If desired, this can easily be avoided by setting Θ_{Li} and Θ_{Ui} to be extremely large in magnitude, but still finite.

4.4. Sketch of identification strategy. The identification strategy uses the fact that Assumption 6** implies Assumption 6 given the other conditions of the identification result, per Lemma 1. Therefore, this sketch uses Assumption 6.

Under Assumption 3 (Optimal strategy is used), for any valuation θ_i, any action $\tilde{a}_i(\theta_i)$ used by player i solves the utility maximization problem in Equation 2, so

$$\theta_i E_{\Pi_i}(\overline{x}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) + \varepsilon_i \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)\right), \tag{13}$$

with $\varepsilon_i = 0$. The reason this sketch allows for non-zero ε_i is explained in Remark 5.

Under Assumption 4 (Correct beliefs), Equation 13 implies

$$\theta_i E_P(\overline{x}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) + \varepsilon_i \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)\right). \tag{14}$$

From the previous discussion of Assumption 5 (Weakly increasing strategy is used), for any $a_i^* \in \mathcal{A}_i$ there is a convex set (in the sense of being a convex subset of the ordered set of valuations)

$$\Theta_i(a_i^*) = \{\theta_i : a_i(\theta_i) = a_i^*\} \tag{15}$$

of valuations such that player i with valuation θ_i uses action a_i^* if and only if $\theta_i \in \Theta_i(a_i^*)$. Moreover, if $a_i \neq a_i'$ then $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are disjoint, given that $\theta_i \in \Theta_i(a_i)$ means $a_i(\theta_i) = a_i$ and $\theta_i' \in \Theta_i(a_i')$ means $a_i(\theta_i') = a_i'$, so $a_i \neq a_i'$ implies it must be that $\theta_i \neq \theta_i'$. Further, if $a_i < a_i'$ and $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are both non-empty then $\sup \Theta_i(a_i) \leq \inf \Theta_i(a_i')$, given that $\theta_i \in \Theta_i(a_i)$ implies $a_i(\theta_i) = a_i$ and $\theta_i' \in \Theta_i(a_i')$ implies $a_i(\theta_i') = a_i'$, so by monotonicity of $a_i(\cdot)$ it must be that $\theta_i < \theta_i'$.

Therefore, for any z_i and $z_i' \in \mathcal{A}_i^d$,

$$E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{x}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{x}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')) \tag{16}$$

$$E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{t}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')). \tag{17}$$

Hence, the beliefs expressions in Equation 14 conditioning on θ_i are generically not point identifiable, because generically multiple valuations use any given $z_i' \in \mathcal{A}_i$.

Equation 14 implies, under Assumptions 4 (Correct beliefs), 5(a) (Weakly increasing strategy is used), and 6 (Monotone effect of counterfactual beliefs on utility), for $\theta_i' < \theta_i < \theta_i''$, for any $z_i \in \mathcal{A}_i$,

$$\theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i') + \varepsilon_i \tag{18a}$$

$$\geq \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) + \varepsilon_i \tag{18b}$$

$$\geq \max_{z_i \in \mathcal{A}_i} \Big(\theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)\Big) \tag{18c}$$

$$\geq \sup_{z_i \in \mathcal{A}_i} (\theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'') - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'')) \tag{18d}$$

$$\geq \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'') - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i''). \tag{18e}$$

In Equation 18, Assumption 4 is used to substitute between E_P and E_{Π_i}. The first step is an implication of Assumption 6(a), since $\theta_i' < \theta_i$. The second step is Equation 14. The third step is an implication of Assumption 6(b), since $\theta_i'' > \theta_i$. The fourth step uses the definition of supremum.

Then, applying Equation 18, and using Assumption 5 (Weakly increasing strategy is used) via the Θ_i construction, for any $z_i \in \mathcal{A}_i$, and any $z_i' < a_i(\theta_i) < z_i''$ with $\{z_i', z_i''\} \in \mathcal{A}_i^d$:

$$\begin{aligned}
&\theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') + \varepsilon_i \\
&= \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) + \varepsilon_i \\
&\geq \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_{P_i}(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) + \varepsilon_i \\
&\geq \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) \\
&= \theta_i E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'').
\end{aligned} \tag{19}$$

Specifically, the first and fourth steps are implications of Equations 16 and 17, relying on Assumption 5. The second step is an implication of the inequality between Equations 18a and 18b. The third step is an implication of the inequality between Equations 18b and 18e. And consequently,

$$\begin{aligned}
\theta_i &\geq \frac{E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - \varepsilon_i - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\
&\quad \forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d, \\
&\qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\} \\
\theta_i &\leq \frac{E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - \varepsilon_i - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \\
&\quad \forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d, \\
&\qquad z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}
\end{aligned} \tag{20}$$

Restricted to specifications with game-structure identification of differences, it follows that

$$\theta_i \geq \frac{E_P(\bar{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - \varepsilon_i - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')} \tag{21}$$

$$\forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\}$$

$$(a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i$$

$$\theta_i \leq \frac{E_P(\bar{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - \varepsilon_i - E_P(\bar{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')}$$

$$\forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}$$

$$(a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i$$

Consequently, the valuation corresponding to a_i must be between $\Phi_{Li}^{(1)}(a_i)$ and $\Phi_{Ui}^{(1)}(a_i)$ from Equations 3 and 5. The other components in the expressions of Equations 4 and 6 are established in the proof of Theorem 1. By another application of Assumption 5 (Weakly increasing strategy is used), any valuation consistent with a_i is between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$. Therefore, $(\Upsilon_{L1}(A_1), \Upsilon_{L2}(A_2), \ldots, \Upsilon_{LN_1}(A_{N_1})) \leq (\theta_1, \theta_2, \ldots, \theta_{N_1}) \leq (\Upsilon_{U1}(A_1), \Upsilon_{U2}(A_2), \ldots, \Upsilon_{UN_1}(A_{N_1}))$, with the inequality viewed element-wise. This is the characterization of the usual multivariate stochastic order mentioned in the discussion of Definition 3.

The identification strategy explores the identification power of the monotone equilibrium assumption, simultaneously allowing for a variety of complications discussed in the introduction (e.g., dependent valuations, "flat spots" in the strategy, non-smoothness, etc.). The special case of Assumption 1* (Independent valuations) is somewhat "special" because it has often been treated separately in the related literature (on auctions, in particular). And, as it turns out, the *statement* of the identification result simplifies in the special case of Assumption 1* (Independent valuations), in the sense that the conditioning on valuations in Equations 13 and 14 and the conditioning on observed actions in Equations 19 to 21 can be dropped. However, as discussed further in Remark 6, the actual *numerical value* of the identified bounds is basically unaffected by the use of Assumption 1*. This reinforces the fact that the same identification strategy applies "equally" to both cases.

4.5. Sharpness.

4.5.1. Independent valuations. Under Assumption 1* (Independent valuations), the identification result in Theorem 1 is "nearly sharp" in the sense formalized by the following result. The definition of "nearly sharp" is discussed in more detail in Remark 10.

Theorem 2. *Suppose that:*

(I) *For all $i \in \mathcal{J}$, $E_P(\overline{x}_i(z_i, A_{-i}))$ and $E_P(\overline{t}_i(z_i, A_{-i}))$ are point identified for $z_i \in \mathcal{K}_i \supseteq \mathcal{A}_i^d$.*

Then there is at least one specification of $E_P(\overline{x}_i(z_i, A_{-i}))$ and $E_P(\overline{t}_i(z_i, A_{-i}))$ for $z_i \notin \mathcal{K}_i$ such that, if it holds that:

(II) *Assumptions 2 and 8(b) hold.*

(III) *For all $i \in \mathcal{J}$, if $a_i \in \mathcal{A}_i^d$ and $z_i \in \mathcal{K}_i$ is such that $E_P(\overline{x}_i(a_i, A_{-i})) = E_P(\overline{x}_i(z_i, A_{-i}))$, then $E_P(\overline{t}_i(a_i, A_{-i})) \leq E_P(\overline{t}_i(z_i, A_{-i}))$.*

(IV) *Actions of different players are independent, so that $P(A) = P_1(A_1)P_2(A_2) \cdots P_N(A_N)$.*

(V) *For all $i \in \mathcal{J}$, $\Gamma_i(\cdot)$ defined on $\mathcal{A}_i^d$ is a strictly increasing function such that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$.*

Then there is a distribution of θ that is marginally equal to the distribution of valuations $(\Gamma_1(A_1), \Gamma_2(A_2), \ldots, \Gamma_{N_1}(A_{N_1}))$ that is such that in the game with that specification of the allocation and transfer rule, there are corresponding weakly increasing strategies resulting in the same distribution of actions as $P(A)$, and such that Assumptions 1 (Independent valuations), 3 (Optimal strategy is used), 4 (Correct beliefs), 5 (Weakly increasing strategy is used), 7 (Known bounds on valuations), 8 (Known bounds on actions) are satisfied.*

Theorem 3. *Under the assumptions used for the independent valuations result in Theorem 1, Assumptions I of Theorem 2 to IV of Theorem 2 hold and there is at least one specification of $\Gamma_i(\cdot)$ that satisfies Assumption V of Theorem 2. Moreover, for $i \in \mathcal{J}$, as long as Assumption 7 holds for finite specifications of Θ_{Li} and Θ_{Ui}, for any $\epsilon > 0$ there are such $\Gamma_i(\cdot)$ with the further property that $0 \leq \sup_{a_i \in \mathcal{A}_i^d} (\Gamma_i(a_i) - \Gamma_{Li}(a_i)) < \epsilon$ and there are such $\Gamma_i(\cdot)$ with the further property that $0 \leq \sup_{a_i \in \mathcal{A}_i^d} (\Gamma_{Ui}(a_i) - \Gamma_i(a_i)) < \epsilon$. Moreover, any distributional property of $F(\theta)$ that is preserved by weakly-increasing component-wise transformations is also a property of the distribution of valuations $(\Gamma_1(A_1), \Gamma_2(A_2), \ldots, \Gamma_{N_1}(A_{N_1}))$ from Theorem 2.*

The assumptions of the identification analysis are sufficient conditions for Theorem 2, per Theorem 3. The reason for writing Theorem 2 in this way is explained below.

Theorem 2 establishes that elements of the identified set from Theorem 1 under Assumption 1* (Independent valuations) indeed do satisfy the assumptions of the identification analysis, per the conclusion of Theorem 2. If $N_1 = N$ in the assumptions, this would be a corresponding Bayesian Nash equilibrium; otherwise, only players 1 through N_1 would satisfy those assumptions.

Relevant when $N_1 < N$, Theorem 2 involves a distribution of the entire vector θ, even though only $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ is bounded in Theorem 1. Theorem 2 takes the displayed distribution of $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ from the identified set from Theorem 1 and finds a joint distribution of the entire vector θ that is marginally equal to the displayed distribution of $(\theta_1, \theta_2, \ldots, \theta_{N_1})$. It does this because some of the assumptions are only sensible in relation to a distribution of the entire vector θ. For example, checking whether Assumption 3 (Optimal strategy is used) is true requires a full specification of a distribution of θ in order to construct the distribution of actions of players $-i$ from the perspective of player i.

Theorem 2 allows that some parts of the expected allocation rule and expected transfer rule may remain unknown to the econometrician even after observing the data. This would happen if the rules are *ex ante* unknown to the econometrician, and certain actions are never used in the observed data. Theorem 2 establishes that there is at least one specification of the expected allocation rule and expected transfer rule (basically, a specification that "fills in" what is both *ex ante* unknown and not identifiable from the data) such that the result described above obtains for the game with that specification of the rules. In the special case that all parts of the expected allocation rule and expected transfer rule are point identified after observing the data (or are known *ex ante*), $\mathcal{K}_i = \mathcal{A}_i$.

In short, Theorem 2 can be used in two main ways.

Theorem 2 can be used to establish "near sharpness" of the identified set from Theorem 1 under Assumption 1* (Independent valuations), by using Theorem 3 to establish the sufficient conditions for Theorem 2, thereby implying that elements of the identified set from Theorem 1 indeed do satisfy the assumptions of the identification analysis. See Remark 10 for an explanation of "nearly sharp."

Theorem 2 can also be used as a set of minimal sufficient conditions for using the identified set. If Assumptions I of Theorem 2 to IV of Theorem 2 are satisfied, the distributions from the identified set from Theorem 1 considered by Assumption V of Theorem 2 indeed do satisfy the assumptions of the identification analysis. This is similar to the use of non-emptiness of an identified set as evidence for correct specification. Since Assumptions I of Theorem 2 to IV of Theorem 2 do not "directly" concern the valuations, they do not appear in the expression for the identified set from Theorem 1. Assumption III of Theorem 2 requires that if two actions have the same expected allocation, and one of them is used in the data, then that action has a weakly better transfer compared to the other action. Assumption IV of Theorem 2 requires that actions are independent across players. Assumption V of Theorem 2 is the condition that the identified set is non-empty, and indeed contains strictly increasing functions between the functions $\Gamma_{Li}(\cdot)$ and $\Gamma_{Ui}(\cdot)$.

Remark 10 (The sense of "nearly sharp")**.** If the lower bound function $\Gamma_{Li}(\cdot)$ (or the upper bound function $\Gamma_{Ui}(\cdot)$) is the same for two or more actions, then the distribution of valuations that is exactly the lower bound (or the upper bound) from Theorem 1 may not be an element of the sharp identified set, as per Theorem 2. This is because the construction in the proof of Theorem 2 would require that those two or more actions are used by a single valuation, which is inconsistent with the assumption of pure strategies as in Assumption 5(a) (Weakly increasing strategy is used). However, per Theorem 3, the lower bound distribution and the upper bound distribution can be approached arbitrarily closely, and thus are "limit points" of the sharp identified set. In that sense, the identified set from Theorem 1 under Assumption 1* (Independent valuations) is "nearly sharp."

4.5.2. ***Dependent valuations.*** Sharpness under Assumption 1 (Dependent valuations) is more complicated. Under Assumption 1 (Dependent valuations), the identification result in Theorem 1 appears to not be "(nearly) sharp." It does not seem possible to directly extend Theorem 2 to the case of Assumption 1 (Dependent valuations). Further, it appears quite difficult to provide a useful[15] characterization of the sharp identified set with dependent valuations, as a consequence of the need to bound player beliefs.

Nevertheless, the identification result under Assumption 1 (Dependent valuations) has two features that are related to "sharpness."

First, the identification result is "sharp in the limit" in the sense that it limits to point identification when the action space either is or limits to a continuous/interval action space, alongside other

[15]Of course, it is always possible to trivially write down the identified set by its definition that it is a distribution of valuations consistent with the data and the assumptions. This might not be particularly useful in empirical practice, since it would presumably require the econometrician to compute a monotone Bayesian Nash equilibrium for every possible distribution of valuations, and check whether it matches the distribution of observed actions. The bounds in Theorem 1 have a closed-form expression that is simple to implement empirically.

conditions, per Section 4.6 and Appendix A. The singleton identified set associated with a point identification result is necessarily the sharp identified set.

Second, and the main focus of this section, the identified set in Theorem 1 is an "approximately sharp" identified set. Using the standard definition, the "sharp" identified set is exactly the set of parameter values that: (a) satisfy all the assumptions used in the identification analysis, and (b) are consistent with the observed data.

This section proposes a notion of an "approximately sharp" identified set. The proposed definition of an "approximately sharp" identified set alters part (a) of the definition of the "sharp" identified set. By the proposed definition, an "approximately sharp" identified set is a set of parameter values that: (a') at least "approximately" satisfy all the assumptions used in the identification analysis, and (b) are consistent with the observed data.

In turn, this requires a definition of at least "approximately" satisfying all the assumptions. In this setting, this is defined to mean that players at least "approximately" maximize utility, thereby allowing for a deviation from Assumption 3 (Optimal strategy is used). Essentially, this corresponds to an ϵ-equilibrium, as in Radner (1980).

Based on these definitions, the following result shows that the identified set from Theorem 1 is "approximately sharp." In order to simplify the statement of the result, define

$$\chi_i(a_i, z_i) = E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i) \text{ and } \tau_i(a_i, z_i) = E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i) \tag{22}$$

Theorem 4. *Suppose that:*

(I) *For all $i \in \mathcal{J}$, $\overline{x}_i(z_i, a_{-i})$ and $\overline{t}_i(z_i, a_{-i})$ are point identified for $(z_i, a_{-i}) \in \mathcal{K}_i = \mathcal{K}_i^i \times \mathcal{K}_i^{-i}$ where $\mathcal{K}_i^i \supseteq \mathcal{A}_i^d$ and $\mathcal{K}_i^{-i} \supseteq \mathcal{A}_{-i}^d$.*

Then there is at least one specification of $\overline{x}_i(z_i, a_{-i})$ and $\overline{t}_i(z_i, a_{-i})$ for $(z_i, a_{-i}) \notin \mathcal{K}_i$ such that, if it holds that:

(II) *Assumptions 2 and 8(b) hold.*

(III) *For all $i \in \mathcal{J}$, $\Upsilon_i(\cdot)$ defined on $\mathcal{A}_i^d$ is a strictly increasing function such that $\Upsilon_{Li}(\cdot) \leq \Upsilon_i(\cdot) \leq \Upsilon_{Ui}(\cdot)$.*

Then:

(a) *There is a distribution of θ that is marginally equal to the distribution of valuations $(\Upsilon_1(A_1), \Upsilon_2(A_2), \ldots, \Upsilon_{N_1}(A_{N_1}))$ that is such that in the game with that specification of the allocation and transfer rule, there are corresponding weakly increasing strategies resulting in the same distribution of actions as $P(A)$, and such that Assumptions 1 (Dependent valuations), 4 (Correct beliefs), 5 (Weakly increasing strategy is used), 7 (Known bounds on valuations) and 8 (Known bounds on actions) are satisfied and the amount of utility foregone by player $i \in \mathcal{J}$ with valuation $\Upsilon_i(a_i)$ for some $a_i \in \mathcal{A}_i^d$ is no more than* $\sup_{z_i \in \mathcal{K}_i^i, z_i \neq a_i} \inf_{\{z_i', z_i''\} \in \mathcal{Z}_i(a_i, z_i)} \Big(-\Big[\Upsilon_i(a_i)\big[\chi_i(a_i, a_i) - \chi_i(a_i, z_i') - [\chi_i(z_i, a_i) - \chi_i(z_i, z_i'')]\big] - \big[\tau_i(a_i, a_i) - \tau_i(a_i, z_i') - [\tau_i(z_i, a_i) - \tau_i(z_i, z_i'')]\big]\Big]\Big)$, *where* $\mathcal{Z}_i(a_i, z_i) = \{\{z_i', z_i''\} \in \mathcal{A}_i^d : z_i' < a_i < z_i'' \text{ and } \chi_i(a_i, z_i') \neq \chi_i(z_i, z_i'')\}$, *using the expressions in Equation 22.*

(b) *Further, if it is known that player $i \in \mathcal{J}$ only considers actions in the set $\tilde{\tilde{\mathcal{A}}}_i$, then the outer* sup *on foregone utility can be taken over the set $z_i \in \mathcal{K}_i^i \cap \tilde{\tilde{\mathcal{A}}}_i, z_i \neq a_i$, with the interpretation being that foregone utility is only relative to the actions in $\tilde{\tilde{\mathcal{A}}}_i$.*

(c) *Further, suppose that $\mathcal{K}_i = \mathcal{A}$ in Assumption* I *of* Theorem 4. *Suppose that all players use weakly increasing strategies, including any $i \notin \mathcal{J}$. Suppose that Assumption* 6**(a) *holds. Suppose that, for each $i \in \mathcal{J}$, $a_{Li}(\cdot)$ and $a_{Ui}(\cdot)$ from Assumption 8 (Known bounds on actions) are such that if a_i and v_i are such that $a_i \in [a_{Li}(v_i), a_{Ui}(v_i)]$, then $a_i \in \tilde{\mathcal{A}}_i(v_i)$ from Assumption* 6**(a). *Suppose either Assumption* 6**(c) *or Assumption* 6**(d) *holds. Then, it also holds that Assumption 6 (Monotone effect of counterfactual beliefs on utility) is satisfied for the stated distribution of valuations and corresponding strategies.*

Theorem 5. *Under the assumptions used for the dependent valuations result in Theorem 1, Assumption* II of Theorem 4 *is satisfied and there is at least one specification of $\Upsilon_i(\cdot)$ that satisfies Assumption* III of Theorem 4. *Moreover, for $i \in \mathcal{J}$, as long as Assumption 7 holds for finite specifications of Θ_{Li} and Θ_{Ui}, for any $\epsilon > 0$, there are such $\Upsilon_i(\cdot)$ with the further property that* $0 \leq \sup_{a_i \in \mathcal{A}_i^d} (\Upsilon_i(a_i) - \Upsilon_{Li}(a_i)) < \epsilon$ *and there are such $\Upsilon_i(\cdot)$ with the further property that* $0 \leq \sup_{a_i \in \mathcal{A}_i^d} (\Upsilon_{Ui}(a_i) - \Upsilon_i(a_i)) < \epsilon$. *Moreover, any distributional property of $F(\theta)$ that is preserved by weakly-increasing component-wise transformations is also a property of the distribution of valuations* $(\Upsilon_1(A_1), \Upsilon_2(A_2), \ldots, \Upsilon_{N_1}(A_{N_1}))$ *from Theorem 4.*

Theorems 4 and 5 basically parallel Theorems 2 and 3. Therefore, much of the logic and discussion in this section builds on Section 4.5.1, and it suffices to discuss the differences. The main difference is that Theorem 4 establishes an upper bound on "foregone" utility, whereas Theorem 2 establishes Assumption 3 (Optimal strategy is used). Consequently, Theorem 4 establishes that the identified set is "approximately sharp," because it establishes that the players at least "approximately" satisfy all the assumptions. Specifically, Theorem 4 establishes that Assumptions 1 (Dependent valuations), 4 (Correct beliefs), 5 (Weakly increasing strategy is used), 7 (Known bounds on valuations) and 8 (Known bounds on actions) are satisfied per Theorem 4(a), and Assumption 6 (Monotone effect of counterfactual beliefs on utility) is satisfied per Theorem 4(c). It establishes that Assumption 3 (Optimal strategy is used) is "approximately" satisfied by providing the upper bound on "foregone" utility.

"Foregone" utility is the difference between the *ex interim* expected utility actually achieved by a player, given its strategy and valuation, and the maximal amount of utility it could have achieved. It takes as given the strategies of the other players, and imposes that players have correct beliefs. In a Bayesian Nash equilibrium, foregone utility is 0 for all players and all valuations.

The bound on foregone utility from Theorem 4 involves point identified quantities, so it can be computed by the econometrician, and is "likely" to be small. This is demonstrated in the numerical illustration in Section 5. It involves a sequence of terms like $E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i) - E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$, which only differ in the value of the conditioning variable. Due to the inf over z_i' and z_i'', such terms are "likely" to be small, when for example $z_i' \approx a_i$ is selected. Under Assumption 1* (Independent valuations), actions are independent across players, in which case the expression in the bound on foregone utility would be 0, basically recovering the result of Theorem 2.

The last part of Theorem 5 implies, for example, that affiliation of $F(\theta)$ implies affiliation of the displayed distribution of valuations from the identified set since affiliation is preserved under monotone mappings, even without using the assumption of affiliation in the identification analysis.

Remark 11 (Establishing Assumption 6). Assumption 8 (Known bounds on actions) plays a role in Theorem 4(c), to establish that Assumption 6 (Monotone effect of counterfactual beliefs on utility) is satisfied via Lemma 1. For example, in a first-price auction, this can be used to eliminate "pathological" strategies where some players bid above their valuations, knowing they will lose for sure. Such strategies could violate Assumption 6. If the definition of "approximately sharp" ignores establishing Assumption 6, this is irrelevant.

Remark 12 (Improving the bound on foregone utility)**.** This result does not meaningfully bound the foregone utility associated with a player with valuation $\Upsilon_i(a_i)$ when a_i is either the smallest used action or largest used action. This is related to the previous discussion of those actions in Remark 6, and holds because the inner inf would be over an empty set in that case. However, the upper bound on foregone utility in Theorem 4 is not necessarily the "best possible" upper bound. For example, in many games, including most auctions, the "best possible" utility a player with valuation v_i can achieve is v_i (i.e., when it cannot do better in the game than getting the object "for free"), and the "worst possible" utility a player with valuation v_i can achieve when it uses the action a_i is $-a_i$ (i.e., when it gets 0 allocation of the object, and fully "transfers" its action). Therefore, in such games, a player that has valuation v_i and uses action a_i foregoes no more than $v_i + a_i$. Although that is generally a poor upper bound, it does improve upon the upper bound being ∞ in the previously mentioned cases. That upper bound is "more informative" in particular for players with low valuations that use low bids. Such arguments can be used to improve the upper bound on foregone utility. That would help to show that the identified set is "approximately sharp" per the overall idea of this section.

4.6. Results with a continuous part of the action space, or increasing number of actions. Consider the limit as $z_i \to a_i, z_i' \uparrow a_i, z_i'' \downarrow a_i$ in the right sides of Equations 3 to 6. This limit can arise when the action space has a continuous part, and the action a_i is in the interior of the action space. Also, this limit can approximate a (heuristic) limit when the number of actions increases to the limit of a continuous/interval action space, with the caveat that the game itself changes when the action space changes, so such a limit cannot be taken literally without a careful analysis of how the game changes. Intuitively, alongside other conditions, point identification can arise under this limit.

A formal point identification result with an interval action space is provided in Appendix A. Note that this point identification requires *more* than just a continuous action space; in particular, it requires the use of *strictly* increasing strategies per Assumption 9. It also requires certain smoothness and differentiability conditions, per Assumptions 10 to 13. As noted in the introduction, it is important that the partial identification strategy accommodates relaxations of those conditions, beyond the conditions on the action space.

A sketch of the intuition for how point identification can arise in the limit goes as follows. Note that $\frac{E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i')-E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')}{E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i')-E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')} \overset{z_i'\uparrow a_i,\ z_i''\downarrow a_i}{\longrightarrow} \frac{\frac{E_P(\bar{t}_i(a_i,A_{-i})|A_i=a_i)-E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{a_i-z_i}}{\frac{E_P(\overline{x}_i(a_i,A_{-i})|A_i=a_i)-E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{a_i-z_i}} \overset{z_i\to a_i}{\longrightarrow} \frac{\left.\frac{\partial E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}$. The first limit requires continuity of the conditional expectations as a function of the conditioning variable, so that $E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i')\to E_P(\bar{t}_i(a_i,A_{-i})|A_i=a_i)$ and $E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i')\to E_P(\overline{x}_i(a_i,A_{-i})|A_i=a_i)$ as $z_i'\uparrow a_i$ and $E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')\to E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)$ and $E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')\to E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)$ as $z_i''\downarrow a_i$, where the third and fourth limits must hold uniformly over z_i since z_i is part of the limiting sequence.[16] The second limit is an application of the definition of the derivative, and requires that the derivatives exist and that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}\neq 0$. In that case, the valuation θ_i corresponding to action a_i is bounded above and below by, and thus must equal, $\frac{\left.\frac{\partial E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}=\Psi_i(a_i)$.[17]

In particular, this suggests that relatively finer discrete action spaces (e.g., auctions that allow bids that are any integer multiple of one cent compared to any integer multiple of five dollars) can be expected to result in relatively tighter identification of the distribution of valuations. This is seen in the numerical illustration in Section 5. However, point identification requires further conditions, beyond just a continuous action space, as discussed in Appendix A.

[16]Continuity of the conditional expectations is related to the smoothness conditions used in Appendix A. Suppose $a_i(\theta_i)=a_i^*$ has the unique solution θ_i^*, so θ_i^* is the unique valuation to use action a_i^*. Then, assuming a density for the valuations, there will be no point mass at a_i^* in the distribution of A_i. Suppose further that $a_i(\cdot)$ is *strictly* increasing in a neighborhood of θ_i^*, and that $a_i(\cdot)$ is continuous in a neighborhood of θ_i^*. The first condition is slightly stronger than the condition that θ_i^* is the unique valuation to use action a_i^*, since it could otherwise be that, for example, $a_i(\cdot)$ is strictly increasing "below" θ_i^*, has a jump discontinuity at θ_i^*, and is flat "above" θ_i^*. Since $a_i(\cdot)$ is weakly increasing per Assumption 5, $a_i(\cdot)$ is continuous except for a countable set. Then, for example, $E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i')=E_P(\bar{t}_i(a_i,A_{-i})|\theta_i=a_i^{-1}(z_i'))$. Supposing that $E_P(\bar{t}_i(a_i,A_{-i})|\theta_i)=E_{\Pi_i}(\bar{t}_i(a_i,a_{-i})|\theta_i)$ is itself continuous as a function of θ_i, it would follow that $E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i')\to E_P(\bar{t}_i(a_i,A_{-i})|A_i=a_i)$ as $z_i'\to a_i$ and similarly for the other limits of the other conditional expectations. Otherwise, if there were multiple valuations to use action a_i, resulting in a point mass at a_i, a "small change" in conditioning on $A_i=a_i$ versus $A_i=z_i'$ could result in a "large change" in the actual expected value, since it would correspond to a "large change" in the set of θ_i being equivalently conditioned on.

[17]This heuristic analysis also implicitly assumes game-structure identification on the right side of Equations 3 to 6. Further, under the condition that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}\neq 0$, assume that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i}$ is continuous in z_i (i.e., continuously differentiable). Consider the case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i}>0$ on an interval neighborhood of a_i. The case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i}<0$ on an interval neighborhood of a_i would be similar, though it seems inconsistent with Assumption 5. Then $E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)$ would be strictly increasing at $z_i=a_i$, and hence (when $z_i'\approx a_i\approx z_i''$), $z_i<a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i')-E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')>0$ in the right side of Equation 3 and $z_i>a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i')-E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')<0$ in the right side of Equation 5.

5. Numerical illustration

This section reports three numerical illustrations of the partial identification result (Theorem 1). The first setting is a sealed-bid first-price auction of a single non-divisible object. The second setting is a sealed-bid first-price auction of two units of a non-divisible object. The third setting is a contest. In all cases, both the action space and the distribution of valuations is discrete.

Because the identification result applies to the class of allocation-transfer game, without using details of any specific game, there is essentially no difference in the mechanical details of applying the identification result to these games. Really, the only difference is the numerical value of the identified bounds (holding fixed the true distribution of valuations). As such, there is a detailed discussion of the auction example, which directly applies also to the contest example.

5.1. Auction.

5.1.1. Primitives. The action space $\mathcal{A}_i$ is a discrete evenly-spaced grid between 0 and 1, for varying numbers of grid points. It is without loss of generality for bids to be restricted to be between 0 and 1, since valuations also have this restriction, as described in the next parts of the setup. For example in auctions, this discreteness corresponds to the issue of a real-world auction that requires that bids are a multiple of some increment, and precludes the use of identification strategies based on derivatives.

There are 10 bidders in each auction, who have correlated valuations. Specifically, in each auction, there is a shared realization of $\eta \sim U[0.25, 0.75]$ and private realizations of $\tau_i \sim U[-0.25, 0.25]$, with $\tilde{\theta}_i = \tau_i + \eta$. This structure induces correlation among the valuations across bidders within an auction. Then, actual valuations are a discretized version of $\tilde{\theta}_i$, with 50 possible values.[18] This highlights the fact that the identification result accommodates discrete distributions of valuations. Without the discretization, this specification of valuations was previously used in Li et al. (2002).

The action space and the valuations can be scaled by the same positive constant without changing the fundamentals of the strategic situation. Thus, the action space can be given multiple equivalent concrete interpretations. For example, by scaling everything by 5, an action space with 6 grid points

[18]The discretization begins by choosing 51 evenly spaced grid points between 0 and 1, namely 0, 0.02, 0.04, and so forth. For each such grid point, the corresponding quantile of the distribution of $\tilde{\theta}_i$ is computed. This results in 50 bins of valuations. Each bin of valuations is assigned a "discretized" value that is the quantile in between the end point quantiles. For the b-th bin, which ranges from the $0.02 \times (b-1)$ quantile to the $0.02 \times b$ quantile, the assigned value is the $0.02 \times (b-1) + 0.01$ quantile. Then a value of $\tilde{\theta}_i$ is discretized by determining which bin it is contained within, and assigned the corresponding "discretized" value.

(0, 0.2, 0.4, 0.6, 0.8 and 1) can be interpreted as bids that must be in whole dollars, for an object that is valued at most 5 dollars. By scaling by 50, the same action space can equivalently be interpreted as bids that must be a scalar multiple of 10 dollars, for an object that is valued at most 50 dollars.

This numerical simulation does require generating the data (unlike in a "real" application of the identification result to pre-existing data), and thus requires computing the Bayesian Nash equilibrium. Details of the computation of the Bayesian Nash equilibrium are provided in Remark 13.

Remark 13 (Computation of the BNE)**.** The identification result takes the data as given from a monotone Bayesian Nash equilibrium (or a relaxation per Remark 5), and does not require the computation of the equilibrium. On the other hand, this numerical simulation must construct the Bayesian Nash equilibrium, in order to generate the data to apply to the identification result. The monotone Bayesian Nash equilibrium is found by numerically iterating on a sequence of best responses. This is closely related to best reply dynamics and fictitious play (e.g., see related ideas summarized in Fudenberg and Levine (1998, 2009)). In general, computing a Bayesian Nash equilibrium is known to be quite difficult (e.g., Cai and Papadimitriou (2014)). But, for the purposes of this paper, it is enough that this algorithm converges (to a Bayesian Nash equilibrium) in the particular games relevant here. Overall, the computation starts by constructing a large (but computationally tractable) number of draws from the distribution of valuations $F(\theta)$. This is the basis for computational approximations to all the quantities based on that distribution. The iterative process starts at some initial strategy s_1. At step j, the strategy s_j is "conjectured" to be used by players 2+, and the best response $\tilde{s}_{j+1}$ (for each valuation) of player 1 is computed. Then, a new strategy $\tilde{\tilde{s}}_{j+1} = r_j s_{j-h:j} + (1 - r_j)\tilde{s}_{j+1}$ is computed, where r_j is a vector of weights and $s_{j-h:j}$ is the equally weighted average of s_{j-h} through s_j (for some length of history h, truncated in the obvious way for the first few iterations). Thus, $\tilde{\tilde{s}}_{j+1}$ is a convex combination of corresponding elements of $s_{j-h:j}$ and $\tilde{s}_{j+1}$. (At some level, it would be "reasonable" to set $\tilde{\tilde{s}}_{j+1} = \tilde{s}_{j+1}$, but this seems to have inferior performance, since it "overreacts.") Finally, $s_{j+1}(\theta^*) = \max_{\theta \leq \theta^*} \tilde{\tilde{s}}_{j+1}(\theta)$. Thus, $s_{j+1}(\theta^*)$ is a weakly monotonic "version" of $\tilde{\tilde{s}}_{j+1}$. Then, s_{j+1} is "conjectured" to be used by players 2+, and the iterative process repeats. Allowing for a small numerical tolerance owing in particular to the numerical approximation to the distribution of valuations, it is computational trivially to check whether a candidate strategy that results from this process is a Bayesian Nash equilibrium.

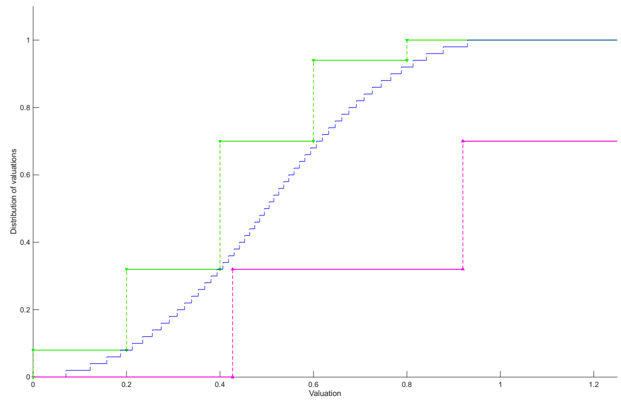
5.1.2. Assumptions. Assumptions 1 (Dependent valuations) and 2 (Action space is ordered) are satisfied trivially, given the setup. Except for a particular alternative analysis in Figure 2c, Assumption 1* (Independent valuations) is not used since this is the dependent valuations case. Assumptions 3 (Optimal strategy is used), 4 (Correct beliefs), and 5 (Weakly increasing strategy is used) are the consequence of the data being generated from the monotone Bayesian Nash equilibrium of this setup, as computed per Remark 13.

Assumption 6**(a) was established in this setting in the discussion above in Example 2, with $\tilde{\mathcal{A}}_i(v_i) = \mathcal{A}_i \cap (-\infty, v_i]$. Assumption 6**(d) holds for the distribution of valuations used.[19] Using this $\tilde{\mathcal{A}}_i(v_i)$, Assumption 6**(e) follows from the construction of the strategy per Remark 13. Thus, Assumption 6** (Monotone *ex post* utility and positively dependent valuations) is satisfied. Consequently, Assumption 6 (Monotone effect of counterfactual beliefs on utility) follows from Lemma 1.

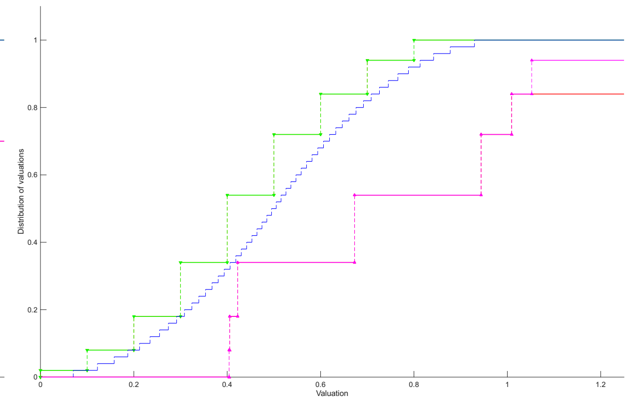
The identification result uses Assumption 7 (Known bounds on valuations), imposing *ex ante* knowledge that valuations are non-negative and that valuations are necessarily no more than 100. This upper bound is much larger than the true upper bound of 1. The specific assumption of 100 has no practical effect (as compared to any number above 1.5, and hence outside the displayed range) over the displayed range of valuations on the horizontal axis in Figure 1. The identification result also uses Assumption 8 (Known bounds on actions) to impose that bids do not exceed valuation.

5.1.3. Results. Each panel in Figure 1 displays the identified bounds for the indicated number of actions. More specifically, each panel concerns the marginal distribution of valuations for any particular bidder (all of whom share the same marginal distribution of valuations, given the setup). The identification result in Theorem 1 also provides identified bounds for the joint distribution of valuations, but this is not possible to display visually. However, it is used in a numerical result later in this section. In each panel in Figure 1, the blue plot is the true cumulative distribution of valuations. The green plot with small "down arrows" is the lower bound for the cumulative distribution of valuations, assuming the econometrician has *ex ante* knowledge of the allocation and transfer rule. The magenta plot with small "up arrows" is the upper bound for the cumulative distribution of valuations, assuming the econometrician has *ex ante* knowledge of the allocation and transfer rule.

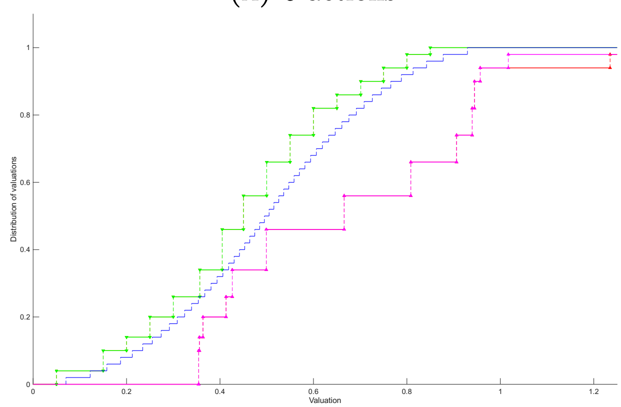
[19] The underlying $\tilde{\theta}$ are affiliated, by construction as used in Li et al. (2002). This footnote concerns the discretization, which can be viewed as applying known weakly increasing functions $g_i(\cdot)$ to $\tilde{\theta}_i$ for each i. Therefore, by Milgrom and Weber (1982, Theorem 3), θ is affiliated.



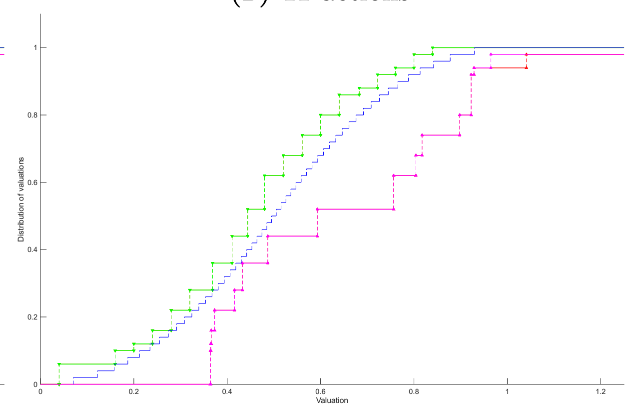
(A) 6 actions



(B) 11 actions



(C) 21 actions



(D) 26 actions

FIGURE 1. Bounds on distribution of valuations, auction.

Given the discreteness in the setup, all of these are actually discontinuous step functions. Each plot also visually includes a series of dashed vertical lines, connecting the different steps. This is only for visual clarity.

Also shown are the lower bound (in orange) and upper bound (in red), using only the identification of the allocation rule and transfer rule following Lemma 2. As expected from Remark 8, the bounds assuming *ex ante* knowledge and the bounds without any *ex ante* knowledge are mostly the same. The only non-zero differences in Figure 1 appear at the right of the figures, for the upper bound. Thus, in fact, the "orange" lower bound is not visible, since it is the same as the "green" lower bound. By construction, whenever the bounds differ, the bounds without *ex ante* knowledge are wider.

As expected, the true distribution of valuations is between the lower bound and the upper bound.

It is possible to use the identified bounds to bound other quantities related to the distribution of valuations. One interesting quantity is the median of the distribution of valuations. Using the bounds from the model with 26 actions, using the bounds with *ex ante* knowledge of the rules, the median of the distribution of valuations is bounded between approximately 0.44 and 0.59. The true median is 0.5. Another interesting quantity is the median of the distribution of the maximum of valuations within each group of 10 bidders. This uses the bounds on the entire joint distribution of valuations, rather than just the bounds on the marginal distribution of valuations. The maximum valuation among the bidders is relevant because it is the transfer the auctioneer would get if the bidder who most valued the object paid its valuation. Equivalently, assuming that the auctioneer has no value for the object, and values the units of the transfers the same way that the bidders do, it is the total welfare in the case that the object is allocated efficiently. Using the bounds from the model with 26 actions, the median of this distribution is bounded between approximately 0.64 and 0.92. The true median of this distribution is approximately 0.71.

In Figure 1a, there are only 6 actions in the action space, and the bounds on the distribution of valuations are relatively wide. With 11 actions in Figure 1b or 21 actions in Figure 1c or 26 actions in Figure 1d, the bounds become increasingly tight. The bounds become much tighter between 6 actions and 11 actions, as compared to between 21 actions and 26 actions. Suppose the maximum possible valuation is interpreted to be 1 dollar. With 6 actions, the difference between the actions is 20 cents (without any scaling). With 11 actions, the difference between the actions is 10 cents. Thus, the gap between the actions halves. On the other hand, with 21 actions, the difference between the actions is 5 cents. And with 26 actions, the difference between the actions is 4 cents, which is not much different from the case of 21 actions.

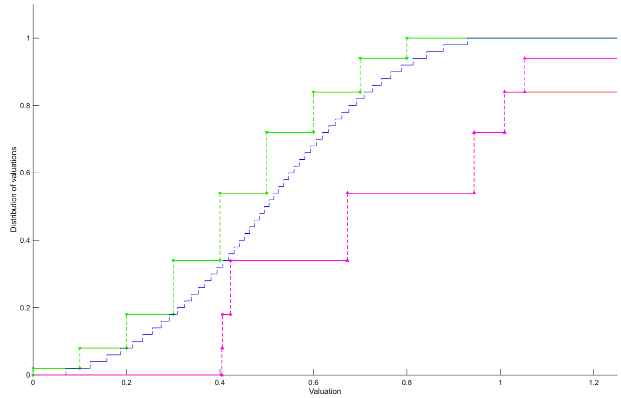
Nevertheless, there is still a "wide" identified set even when bids have 5 cent increments (under 21 actions) or 4 cent increments (under 26 actions). This suggests caution against using the point identification result in this paper (e.g., Theorem 6) or existing point identification results from the literature, unless the conditions of those results are fully correct in the intended application. It may be tempting to use those point identification results "as an approximation" even if the conditions are not fully correct, but this would ignore the partial identification of the distribution of valuations, resulting in empirical results that overstate what can actually be learned from the data. Note also that a continuous action space alone is not sufficient for point identification, as discussed in Section 4.6 and Appendix A, so having a "large number" of actions is not guaranteed to be a good approximation to the assumptions needed for point identification.

In each panel of Figure 1, the upper bound (the magenta plot with "up arrows") never attains the value of 1 *over the displayed range of valuations.* If the displayed range of valuations were extended all the way to 100 (thereby completely obscuring what happens over the "interesting" range that is displayed), it would be seen that the upper bound does eventually attain the value of 1. This is a direct consequence of the identification strategy, as follows. In Theorem 1, the *upper* bound for the valuation consistent with using the largest action *that is used* is the *ex ante* upper bound on valuations.[20] This is because, considering the specification of $\Phi_{Ui}^{(1)}(a_i)$ in Equation 5, there cannot be $z_i'' \in \mathcal{R}_i$ with $a_i < z_i''$, when a_i is the largest action that is used. Thus, any probability mass on the largest action *that is used* translates to an upper bound that is the *ex ante* upper bound on valuations. Similarly, the *lower* bound for the valuation consistent with using the smallest action *that is used* is the maximum of the *ex ante* lower bound on valuations and $a_{Ui}^{-1}(\cdot)$ evaluated at that action.

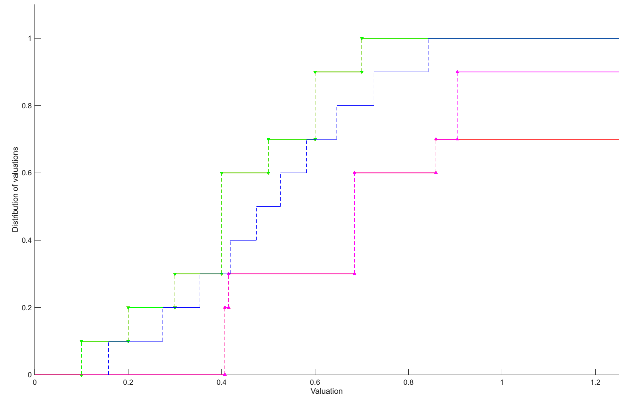
Based on Theorem 4, in the model with 26 actions, using the bounds with *ex ante* knowledge of the rules, and for the distribution of valuations that is the midpoint between the upper bound and lower bound, approximately 92% of the distribution of valuations of player i (for any i, given symmetry) can be given a non-trivial bound on the "foregone" utility. Among those valuations, the average corresponding upper bound on the "foregone" utility is approximately 0.01. When there are only 6 actions, those two numbers are approximately 86% and 0.05, respectively. Consistent with the discussion of Theorem 4, this illustrates that the identified bounds can be expected to "get sharper" with more actions, with the limiting result of point identification in Theorem 6.

Figure 2 shows the bounds for a few alternative specifications, changing one aspect per panel. The main message is that the identified bounds remain "reasonable" across a variety of setups. Unless it is the aspect changed, each panel concerns the same distribution of valuations as described above, with 50 discrete grid points, and 11 actions. This baseline case is Figure 2a, which just repeats Figure 1b for visual reference. Figure 2b shows the results with fewer grid points in the distribution of valuations. Figure 2c shows the results for the case of independent valuations, where each bidder has an independent draw from the same marginal distribution of valuations (basically, each bidder now gets its own independent value of η). Figure 2d shows the results for the case of discretized log-normal distribution of valuations, where underlying multivariate normal random variables with 0.25 variances and covariances 0.025 are drawn and exponentiated, and then scaled by $\frac{1}{2}$ in order to generally fall in the unit interval.

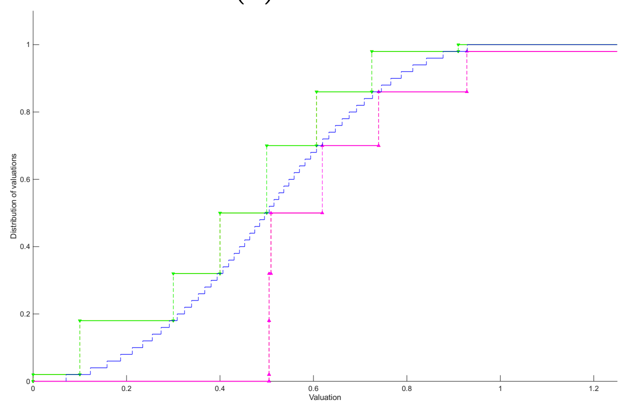
[20]Recall, the upper bound is the minimum of the *ex ante* upper bound on valuations and $a_{Li}^{-1}(\cdot)$ evaluated at that action. In this particular case, the upper bound is not impacted by Assumption 8, as previously discussed.



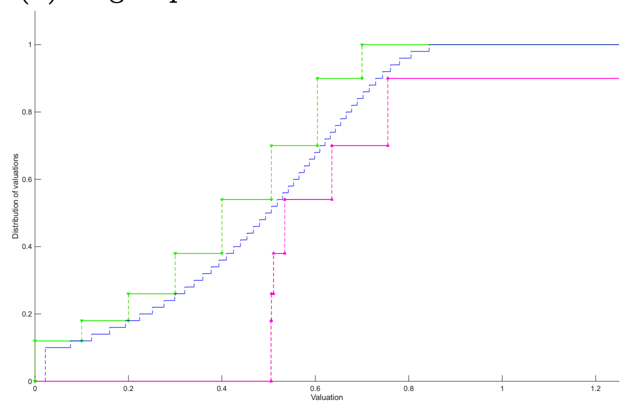
(A) Baseline



(B) 10 grid points in distribution of valuations



(C) Independent valuations



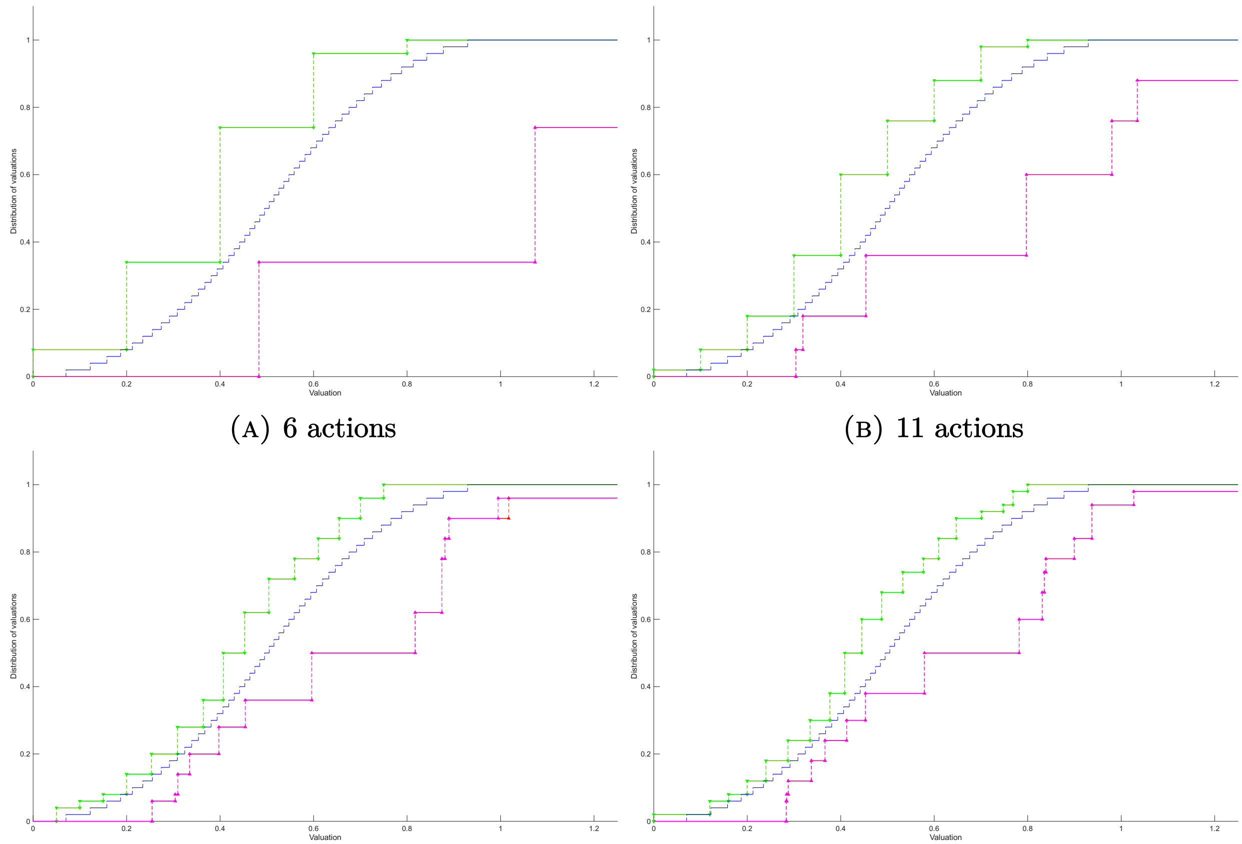
(D) Discrete log-normal valuations

FIGURE 2. Bounds on distribution of valuations, auction. Alternative setups.

5.2. Auction with two objects. The second numerical illustration concerns a first-price auction with two objects, rather than just one object. This changes both the allocation rule and the transfer rule, since now the allocation rule gives the *two* highest bidders one unit of the object each (and randomly assigns in cases of ties, making sure that any bidders with higher bids are allocated any available units before bidders with lower bids).[21] Correspondingly, the transfer rule accounts for the new rule describing when a bidder wins a unit of the object, and thus pays its bid. Because different winners can potentially pay different prices, this is a discriminatory auction. The distribution of valuations and the action space is the same as in the previous auction illustration. Note that the same reasoning for the assumptions as in Section 5.1.2 applies here.

The panels in Figure 3 emulate those in Figure 1. The bounds are basically identical to those in Figure 1. Thus, it seems that multiple units has minimal impact on the "informativeness" of the data about the distribution of valuations. This shows that the identification strategy easily accommodates situations with multiple units.

[21]For example, if the highest three bids are 1, 0.9, and 0.9, the bidder who bids 1 gets a unit for sure, and the other two bidders get a unit with probability 0.5.



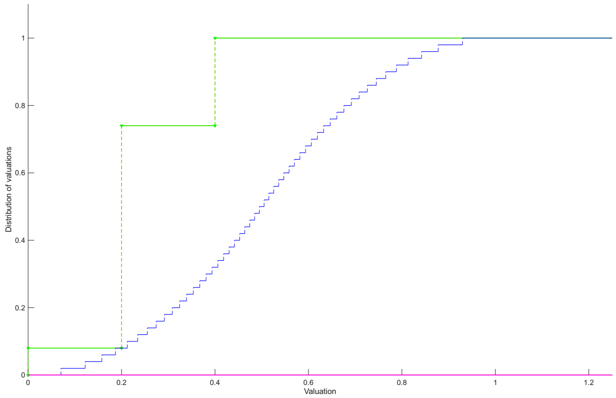
(A) 6 actions (B) 11 actions

(C) 21 actions (D) 26 actions

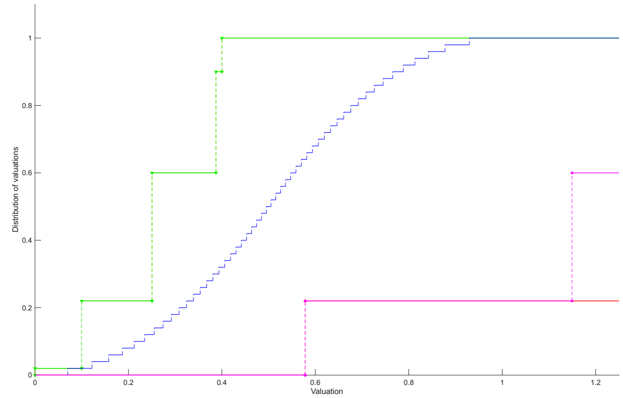
FIGURE 3. Bounds on distribution of valuations, auction with two objects.

5.3. Contest. The third numerical illustration concerns a contest. Compared to the auction illustration, the difference is that now the allocation rule is the “lottery” specification of the contest success function from Example 1. Thus, the probability that a player wins the contest is equal to that player’s relative proportion of the total efforts of all players. This is an example where it is particularly useful that the identification strategy allows the econometrician to use the data to identify the allocation rule, rather than require it be known *ex ante*, since perhaps other contest success functions are also plausible. (This would depend on what institutional knowledge exists in a given empirical application.) The distribution of valuations and the action space is the same as in the auction illustration. The transfer rule is that the “winner” of the contest pays its “effort.” Note that the same reasoning for the assumptions as in Section 5.1.2 applies here, except using the previous reasoning in Example 1 to establish Assumption 6**(a).

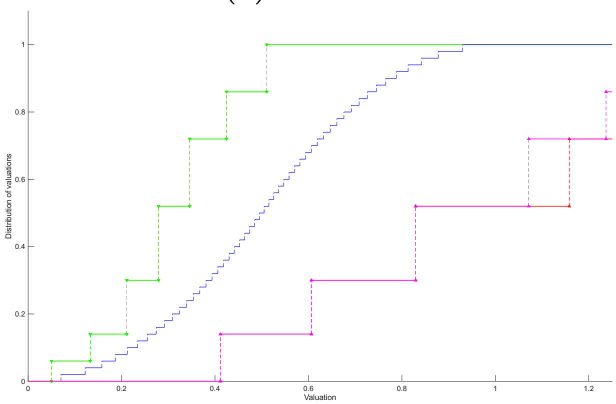
The panels in Figure 4 emulate those in Figure 1, except based on this contest game. The bounds are noticeably wider compared to those in Figure 1. Thus, it seems these contests are “less informative” about the distribution of valuations.



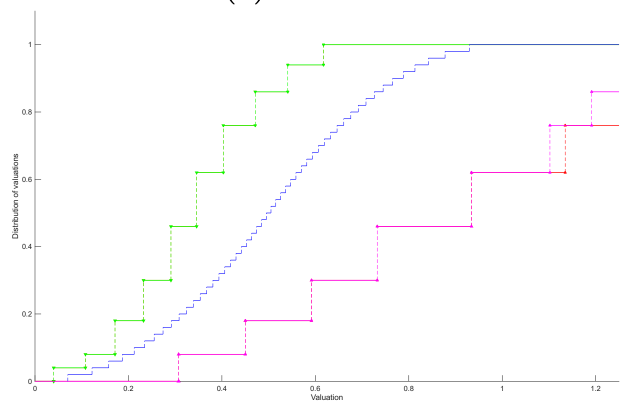
(A) 6 actions



(B) 11 actions



(C) 21 actions



(D) 26 actions

FIGURE 4. Bounds on distribution of valuations, contest.

6. Conclusions

This paper develops identification results for the distribution of valuations in a class of allocation-transfer games that determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The identification results are constructive and are based on the assumption of the use of monotone strategies. The results allow dependent valuations, discrete parts of the action space, non-smoothness, and unknown (to the econometrician) details of how the allocations and transfers are determined.

A. Point identification in the limit

As noted in Section 4.6, the partial identification result "limits" to point identification under certain conditions. This section formalizes that result.

Assumption 9 (Continuous action space and no point masses in distribution of actions)**.** *For each* $i \in \mathcal{J}$, $\mathcal{A}_i = [\alpha_i, \beta_i]$ *and there are no point masses in the observed distribution of actions of player* i.

Compared to Assumption 2 (Action space is ordered), Assumption 9 rules out discrete actions.

Assumption 10 (Smooth distribution of valuations). *The distribution $F(\cdot)$ has associated ordinary density $f(\cdot)$. For each $i \in \mathcal{J}$, the support of the distribution of θ_i is an interval.*

Under Assumptions 1 (Dependent valuations), 5 (Weakly increasing strategy is used), and 10 (Smooth distribution of valuations), lack of point masses from Assumption 9 (Continuous action space and no point masses in distribution of actions) is equivalent to the condition that the strategy is *strictly* increasing.[22] As discussed in the introduction, and as noted for example in Example 1, Example 2, Example 5, and Example 6, this condition may not hold in some games, even with a continuous action space, since strategies can involve "flat spots" in general. Therefore, a continuous action space "by itself" should not be expected to result in point identification. Further, as a smoothness condition on the distribution of valuations, Assumption 10 may be too strong, as discussed in the introduction.

Assumption 11 (Differentiable *ex interim* expected allocation and expected transfer). *For each $i \in \mathcal{J}$, there is a set $\mathcal{E}_{i,d}$ with $P(A_i \in \mathcal{E}_{i,d}) = 0$ such that for each possible valuation θ_i, the expected allocation $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)$ and the expected transfer $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)$ are differentiable functions of a_i, evaluated at any $a_i^* \in support(a_i(\theta_i)) \cap \mathcal{E}_{i,d}^C$.*

The notation S^C for some set S is the complement of the set S. Assumption 11 requires that *ex interim* expected allocation and *ex interim* expected transfer given valuation θ_i are differentiable on the support of the strategy $a_i(\theta_i)$. Intuitively, this corresponds to the existence of the derivatives used in the heuristic argument in Section 4.6. Under Assumption 5 (Weakly increasing strategy is used), $a_i(\theta_i)$ is a degenerate random variable (i.e., a pure strategy). However, under Assumption 3 (Optimal strategy is used) alone, mixed strategies are allowed. As mentioned above, breaking up the assumptions in this way makes it easier to refer to the separate roles of the assumptions. Assumption 11 allows a probability zero exceptional set of actions at which differentiability fails.

Of course, differentiability is too strong with discrete actions. It can also be too strong even in games with a continuous action space, as illustrated most directly in Example 6. In particular, the

[22] If two valuations use the same action, then there is a point mass at that action because the entire interval connecting those valuations would also use that same action. So, if there are no point masses, then no two valuations use the same action, so the strategy must indeed be *strictly* increasing. Conversely, obviously if the strategy is *strictly* increasing, then there are no point masses in the distribution of actions by Assumptions 1 and 10. This conclusion is not true without Assumption 5, since if $a_i(\cdot)$ were non-monotone, then the set $\{\theta_i : a_i(\theta_i) = a_i^*\}$ can be non-singleton, but not necessarily of positive probability under the distribution of θ_i. Therefore, if the strategy were non-monotone, then multiple valuations could use the same action a_i^* even though there is no point mass at a_i^*.

use of strategies with "flat spots" can induce corresponding discontinuities in the *ex interim* expected utility. Differentiability is not needed in the partial identification results.

Let

$$\Psi_i^x(z) \equiv \left.\frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \quad \text{and } \Psi_i^t(z) \equiv \left.\frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \tag{23}$$

and let

$$\Psi_i(z) \equiv \frac{\Psi_i^t(z)}{\Psi_i^x(z)}. \tag{24}$$

The proof of Theorem 6 shows that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ actually do exist for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,d}^C$.

Definition 5 (Game-structure identification of derivatives). An action a_i is an action with game-structure identification of derivatives if $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ can be identified to exist, and $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified quantities. Per convention, identification of derivatives on the boundary of $\mathcal{A}_i$ is understood to concern identification of the corresponding *one-sided* derivative.

Assumption 12 (Game-structure identification of derivatives). *For each $i \in \mathcal{J}$, there is a set $\mathcal{E}_{i,r}$ with $P(A_i \in \mathcal{E}_{i,r}) = 0$ such that if $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,r}^C$ is such that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist then a_i is an action with game-structure identification of derivatives per Definition 5.*

Assumption 12 requires game-structure identification of derivatives for all actions used in the data except for the probability zero exceptional set $\mathcal{E}_{i,r}$. This accommodates the possibility that game-structure identification of derivatives may fail on a set of probability zero. Similar to game-structure identification of differences from Definition 4, game-structure identification of derivatives follows from standard conditions on identification/estimation of derivatives of conditional expectations. See Lemma 3 for details.

Assumption 13 (Non-zero marginal expected allocation). *For each $i \in \mathcal{J}$, there is a set $\mathcal{E}_{i,m}$ with $P(A_i \in \mathcal{E}_{i,m}) = 0$ such that $\Psi_i^x(a_i) \neq 0$ for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,m}^C$.*

Assumption 13 allows a probability zero exceptional set $\mathcal{E}_{i,m}$.

Theorem 6. *Under Assumptions 1 (Dependent valuations), 2 (Action space is ordered), 3 (Optimal strategy is used), 4 (Correct beliefs), 5 (Weakly increasing strategy is used), 9 (Continuous action space and no point masses in distribution of actions), 10 (Smooth distribution of valuations), 11*

(Differentiable ex interim *expected allocation and expected transfer), 12 (Game-structure identification of derivatives), and 13 (Non-zero marginal expected allocation), the distribution of valuations* $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ *is point identified, and the identification is constructive, because the distribution of* $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ *equals the distribution of* $(\Psi_1(A_1), \Psi_2(A_2), \ldots, \Psi_{N_1}(A_{N_1}))$*, where* $(A_1, A_2, \ldots, A_{N_1})$ *is distributed according to the data* $P(A, X, T)$ *and* $\Psi_i(\cdot)$ *is the identifiable function given in Equation 24.*

Independent valuations. *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 1* (Independent valuations) and replace the* Ψ *functions with the* Λ *functions defined in Equation 26.*

Let

$$\Lambda_i^x(z) \equiv \left.\frac{\partial E_P(\overline{x}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z} \text{ and } \Lambda_i^t(z) \equiv \left.\frac{\partial E_P(\overline{t}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z}. \tag{25}$$

Also, let

$$\Lambda_i(z) \equiv \frac{\Lambda_i^t(z)}{\Lambda_i^x(z)}. \tag{26}$$

Then,

$$\Lambda_i^x(z) = \left.\frac{\partial E_P(X_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z} \text{ and } \Lambda_i^t(z) = \left.\frac{\partial E_P(T_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z}. \tag{27}$$

Under Assumption 1 (Independent valuations), the econometrician can point identify* $\Lambda_i^x(\cdot)$ *and* $\Lambda_i^t(\cdot)$ *using the expressions in Equation 27.* ★

The following provides one sufficient condition for game-structure identification of derivatives, formalizing the idea that it follows from standard results on identification and estimation of conditional expectations.

Lemma 3 (Sufficient conditions for game-structure identification of derivatives)**.** *Suppose that Assumptions 1 (Dependent valuations) and 9 (Continuous action space and no point masses in distribution of actions) are satisfied. Let an action* $a_i \in \mathcal{A}_i$ *be given, for some* $i \in \mathcal{J}$*. Suppose* $a_i \in \mathcal{A}_i^d$*, and suppose there is a set* $\mathcal{S}$ *containing* a_i *such that* $\mathcal{A}_i^d \cap \mathcal{S}$ *is a non-degenerate interval and such that* $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ *and* $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ *are point identified for all* $a_i' \in \mathcal{A}_i^d \cap \mathcal{S}$ *and* $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$*, where* $\tilde{\mathcal{A}}_{-i}^d(a_i')$ *has probability* 1 *according to the distribution* $A_{-i}|(A_i = a_i)$*. Suppose* $A_{-i}|(A_i = a_i)$ *is point identified. Suppose: If* $a_i \in int(\mathcal{A}_i)$*, then* $a_i \in int(\mathcal{A}_i^d \cap \mathcal{S})$*. Suppose the data is* $P(A, X, T)$*. Then, whether or not* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *exist is point identified. Exists means,* *by definition, that the limit corresponding to the definition of the derivative exists. Moreover, if* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *exist, then there is game-structure identification of derivatives per Definition 5. Identification of* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *is constructive, and given by the existence and values of the limits corresponding to expressions in Equation 28:*

(28)
$$\Psi_i^x(z) = \frac{\partial E_P(E_P(X_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\bigg|_{a_i=z} \quad \textit{and} \quad \Psi_i^t(z) = \frac{\partial E_P(E_P(T_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\bigg|_{a_i=z}$$

B. Examples of games

The class of allocation-transfer games studied in this paper is illustrated via examples.

Example 1 (Contests, continuing from p. 10)**.** In contest models, the actions are interpreted as "costly effort" toward winning a valuable object. The economic theory of such models has been developed in, for example, Hillman and Riley (1989), Baye et al. (1993), Amann and Leininger (1996), Krishna and Morgan (1997), Lizzeri and Persico (2000), Parreiras and Rubinchik (2010), and Siegel (2014), in addition to an overall large literature. See for example Konrad (2007, 2009) for a summary of the literature, including discussion of theoretical applications to a broad range of instances of competition, including advertising, litigation, political lobbying, research and development, and sports. Among other examples, including some of the above cited literature, Wasser (2013), Ewerhart (2014), Lu and Parreiras (2017), and Prokopovych and Yannelis (2023) establish conditions for a monotone equilibrium.

The valuation θ_i is the value that player i has for the object. Often, the "efforts" are equivalent to financial expenditures, so that $\mathcal{A}_i = [0, \infty)$ and the transfer rule is $\bar{t}_i(a) = a_i$. However, other transfer rules are also possible. For example, it might be that part of the effort is "refundable," so that players only expend some fraction of their effort, possibly depending on whether the player wins or loses (e.g., see the models in Riley and Samuelson (1981) and Matros and Armanios (2009)). The allocation rule $\overline{x}(a) = (\overline{x}_1(a), \overline{x}_2(a), \ldots, \overline{x}_N(a))$ is known as the "*contest success function*" that relates the actions taken by the players to the probabilities that each of the players wins the valuable object. The econometrician may not know the contest success functions $\overline{x}(\cdot)$, and indeed the economic theory literature has explored a variety of different contest success functions. See for example Corchón and Dahm (2010) for a detailed discussion. For example, following Tullock (1980)-style models,

$\overline{x}_i(a) = \begin{cases} \frac{a_i^r}{\sum_{j=1}^N a_j^r} & \text{if } a \neq 0 \\ \frac{1}{N} & \text{if } a = 0 \end{cases}$ for some $r > 0$. In particular, the case of $r = 1$ has been interpreted as a "lottery" in which the probability that player i wins is equal to player i's share of the overall aggregate effort. The specification states that if all players expend no effort, then each player has equal chance of winning the contest. More generally, there can be functions $f_i(\cdot)$ such that $\overline{x}_i(a) = \frac{f_i(a_i)}{\sum_{j=1}^N f_j(a_j)}$, including the logistic specification $f_i(z) = e^{kz}$ as in Hirshleifer (1989). Alternatively, following Lazear and Rosen (1981)- and Dixit (1987)-style models, $\overline{x}_i(a) = P_\epsilon(a_i + \epsilon_i > \max_{j \neq i}(a_j + \epsilon_j))$, where P_ϵ is the distribution of "noise" or "randomness" involved in determining the contest winner. The identification results do not require the econometrician to know $\overline{x}(\cdot)$ (or the underlying distribution $\tilde{x}(\cdot)$). In particular, the econometrician might not know r or f_i or P_ϵ.

In the above specifications, generally a player that expends the most effort is most likely to win, but is not guaranteed to win. In the limiting case of the "all-pay auction" formulation,

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } i \text{ expends the most effort} \\ p_i(a) & \text{if } i \text{ ties for expending the most effort with at least one other player} \\ 0 & \text{if } i \text{ does not expend the most effort,} \end{cases}$$

where $p_i(a)$ reflects the tie-breaking rule. In all-pay auction models, the player that expends the most effort is guaranteed to win.

Contests (and related all-pay auctions) can involve the use of strategies with "flat spots" in which players with low valuations are "inactive" (use zero effort) in the contest in order to avoid "unprofitable" effort, as discussed in Amann and Leininger (1996, page 10), Parreiras and Rubinchik (2010), Lu and Parreiras (2017, Proposition 2), or Ewerhart and Quartieri (2020, page 257).

Example 2 (Auctions, continuing from p. 11)**.** Auction models can involve various complications like "participation costs," reserve prices, asymmetries, and/or multiple units possibly with endogenous supply. The economic theory of auctions has been reviewed, for example, in Klemperer (1999, 2004), Milgrom (2004), Menezes and Monteiro (2005), and Krishna (2009). Specifically the economic theory of auctions with a discrete action space has been developed in Chwe (1989), Rothkopf and Harstad (1994), Dekel and Wolinsky (2003), David et al. (2007). One feature of the auction theory literature is the range of auction formats, implying a range of allocation and transfer rules. Much of the economic theory literature has focused on establishing monotonicity of the strategy in auction models, and moreover the literature on general conditions for monotone equilibrium in games often treats auctions as a leading example of their results. See for example Plum (1992), Lizzeri and Persico (2000), Maskin and Riley (2000a,b, 2003), and Jackson and Swinkels (2005).

The valuation θ_i is the value player i has for a unit of the object being auctioned. The specific auction format would be reflected in the allocation rule $\overline{x}(\cdot)$ and transfer rule $\overline{t}(\cdot)$, and the identification strategy can apply to a wide range of auction formats.

Let $H_i(a) = \max_{j \neq i \text{ and } j \text{ s.t. } a_j \geq r_j} a_j$ be the highest bid other than the bid of player i, among the bids from players that exceed the corresponding reserve price, where $r_i \geq 0$ is the reserve price for player i.

Also let $S(a)$ be the quantity allocated to the winning bidder as a function of the profile of bids (e.g., Milgrom (2004, Section 4.3.3)). For example, the supply $S(a)$ might depend only on the winning bid, as in a "supply curve" at the "price" of the winning bid. See also Example 3 for related models where $S(a)$ can be interpreted as a "demand curve." The standard case that there is one exogenous unit of the object being auctioned is the special case that $S(\cdot) \equiv 1$.

The allocation is the awarding of units of the object from the auction. Then, for example, in auction formats where the highest bidder wins, as long it exceeds its reserve price and the highest competitor's bid among those bids exceeding the corresponding reserve price,

$$\overline{x}_i(a) = \begin{cases} S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i, \end{cases}$$

where $p_i(a) \in [0, S(a)]$ reflects the tie-breaking rule, the expected number of units that player i is allocated when bids are a, involving a tie for high bid.

The transfers include the payments to the auctioneer, but could include other transfers, like participation costs[23] when applicable. The transfer rule also depends on the auction format. For

[23] A participation cost can be modeled in a few different ways, particularly concerning whether the players know their own valuation at the time they make the participation decision. A third approach allows that bidders observe a signal of their valuation at the time of their participation decision, an identification problem studied in Gentry and Li (2014). Other identification results emphasizing entry/participation in particular auction models includes Marmer et al. (2013) (focusing on identifying the selection effect, and discriminating between models of entry), Fang and Tang (2014) (focusing on inferring bidder risk attitudes), and Li et al. (2015) (focusing on testable implications of risk aversion). The economic theory of auctions with participation costs has been developed in, for example, Samuelson

example in a first price auction, and noting that $\bar{t}_i(a)$ is the *expected* transfer that integrates over the tie-breaking rule,

$$\bar{t}_i(a) = \begin{cases} a_i S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ a_i p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i \end{cases}$$

Other auction formats would have different allocation rules and/or transfer rules.

The econometrician may not know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$, because the econometrician may not know the "supply function" $S(a)$. The identification results do not require the econometrician to know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$.

Bidding strategies can involve "flat spots." For example, per the cited literature above, participation costs and/or reserve prices intuitively generate "cutoff strategy" bidding strategies such that all sufficiently low valuations use the "do not participate" action. Even without those features, some equilibria in some auctions can have "flat spots" as in Blume and Heidhues (2004, Proposition 1) or as in the discussion of all pay auctions in Example 1.

Because the allocation-transfer game framework does not necessarily require the assumption of symmetric players, the auction could involve such asymmetries as "strong" and "weak" bidders, as in Milgrom (2004, Section 4.5). For example, Campo et al. (2003) have focused on establishing point identifying assumptions for asymmetric bidders with affiliated private values in first price auctions. Reny and Zamir (2004) have studied the existence of monotone equilibrium in related auction models.

Henderson et al. (2012) and Luo and Wan (2018) explore the impact of monotonicity of the bidding strategy in specific first-price auction models with independent valuations on the properties of the estimator (e.g., rate of convergence, optimality, etc.), whereas this paper explores the role of monotonicity in identification.

Haile and Tamer (2003) study the (partial) identification of bidder valuations that arises when the econometrician has an incomplete model, specifically in an incomplete model of English auctions with symmetric independent private values. See also Chesher and Rosen (2017) for further identification results in a related model, based on generalized instrumental variables. Haile and Tamer (2003) studied identification of bidder valuations based on the assumptions that bidders will not be "outbid" and will not "overbid."

(1985), McAfee and McMillan (1987), Levin and Smith (1994), Tan and Yilankaya (2006), and Cao and Tian (2010). See for example Krishna (2009, Section 2.5) for equilibrium in auctions with reserve prices.

Another important identification problem, also leading to partial identification, particularly in certain auction formats, concerns the "missing data" problem when the econometrician does not observe the bids of all the players. Aradillas-López et al. (2013) have established partial identification in the important case of an ascending auction with correlated valuations, focusing on showing partial identification of economically relevant seller profit and bidder surplus quantities rather than the object in this paper, the overall joint distribution of valuations. Because the data used by the identification strategy developed here includes the actions of all players, it cannot be applied to address the identification problem studied in Aradillas-López et al. (2013). However, the identification strategy developed here does allow "missing data" on other parts of the game, for example the "participation cost" in an auction with a participation cost. Similarly, because the identification strategy can apply to an incomplete specification of the model, the identification results also accommodate "missing *ex ante* knowledge," for example on endogenous quantity functions in an auction. Tang (2011) focuses on partial identification of auction revenue in first-price auctions with common values, which also is not addressed by this paper, which assumes private values.

Example 3 (Procurement auctions, reverse auctions, oligopoly models, etc.)**.** Models of procurement auctions, reverse auctions, and related situations are similar to auctions, with the distinguishing feature that the N players are bidding to *sell* units of an object, rather than *buy* units of an object. Therefore, the valuation θ_i can be interpreted to be player i's (constant) marginal cost of supplying one unit of the object, and the "low bid" wins the market. Let $L_i(a) = \min_{j \neq i \text{ and } j \text{ s.t. } a_j \leq r_j} a_j$ be the lowest bid other than the bid of player i, among the bids from players that are below the corresponding reserve price. The "allocation" experienced by player i is the quantity of the object that player i supplies, and therefore the allocation is negative, so the allocation rule could be

$$\overline{x}_i(a) = \begin{cases} -S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i, \end{cases},$$

where, similarly to Example 2, $S(a)$ is the endogenous quantity (i.e., "demand") given the profile of bids a, r_i is the maximum acceptable bid for player i, and $p_i(a)$ reflects the tie-breaking rule. The "transfer" experienced by player i is the payment to player i. Due to the convention in this paper that transfers are *from* the player, transfers are negative. For example, it could be that

$$\bar{t}_i(a) = \begin{cases} -a_i S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -a_i p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i \end{cases}$$

Some models of oligopoly competition are basically the same game, with N firms in an oligopoly having privately known constant marginal costs of production competing to win the oligopoly market, see for example Vives (2001, Chapter 8). In these models, the "endogenous quantity" $S(a)$ is the demand curve, generally depending on the lowest bid (i.e., the "realized price"). As with the endogenous supply in Example 2, the econometrician may not know the "demand curve" and therefore not know $\bar{x}_i(a)$ and/or $\bar{t}_i(a)$. The identification results do not require the econometrician to know $\bar{x}_i(a)$ and/or $\bar{t}_i(a)$.

Example 4 (Partnership dissolution)**.** The economic theory of partnership dissolution has been developed in Cramton et al. (1987), in addition to a large subsequent literature. There are N co-owners of an object. Prior to partnership dissolution, player i owns share r_i of the object and has valuation θ_i for the object. The econometrician need not know these ownership shares.

In the "bidding game" formulation of partnership dissolution developed in Cramton et al. (1987), there are initial transfers between the co-owners that depend on their ownership shares. Since these initial transfers do not depend on valuations, they are not revealing of valuations. In the special case of equal ownership shares, these initial transfers are zero. In any case, the econometrician need not observe data on these initial transfers in order to apply the identification strategy. Indeed, the identification strategy does not rely on the game implementing such initial transfers. These initial transfers are for purposes of satisfying the individual rationality constraint, violation of which does not change the identification strategy in this paper, since this paper essentially only uses the incentive compatibility constraint. See formula C of Cramton et al. (1987, Theorem 2). Then, each co-owner bids for ownership, so the action in the game are bids, with the highest bidder winning ownership. The transfer from player i is (omitting the "lump sum" initial transfer reflecting ownership shares but not valuations) $\bar{t}_i(a) = a_i - \frac{1}{N-1}\sum_{j \neq i}^{N} a_j$, so player i transfers its bid even if it loses, and is "compensated" by the bids of the other players.

Example 5 (Public good provision)**.** In models of the provision of public goods or public projects, the distinguishing feature is that the allocation is the same to all players, reflecting the "public" nature of the object. The valuation θ_i reflects the private value that player i places on the public good. The economic theory of such models has been developed in Bergstrom et al. (1986), Bagnoli and Lipman (1989), Mailath and Postlewaite (1990), Alboth et al. (2001), Menezes et al. (2001), and Laussel and Palfrey (2003), in addition to a large overall literature, summarized for example in Ledyard (2006). See Lemma 1 or the discussion of "regular" equilibrium in Laussel and Palfrey (2003) for the role of monotonicity in the strategies. Or see the characterization of the equilibrium strategies in Menezes et al. (2001). In direct revelation games (e.g., Clarke (1971)-Groves (1973) games), players report their valuation, in which case the identification problem is trivial. However, in other games, the actions of the players are interpreted as contributions to the public good, and the object is allocated (e.g., the public project is completed) if and only if the sum of the contributions of the players is greater than the cost of producing the public good. The contributions may or may not be refunded if the public good is not produced, depending on the specific game. See for example Menezes et al. (2001). Some models of public good provision, along the lines of Palfrey and Rosenthal (1984) (who worked with complete information), involve a discrete and even binary action space (contribute an *ex ante* fixed amount or not). See for example Laussel and Palfrey (2003, Section 3) for a discussion of step function equilibria with "flat spots" in the strategies, or see for example the equilibria with "flat spots" in Menezes et al. (2001).

Example 6 (Strategic (non-"price taking") market behavior)**.** Models of strategic (non-"price taking") market behavior, specifically models based on multilateral double auctions, involve N_s sellers (i.e., players that currently each own a unit of the object) and N_b buyers (i.e., players that potentially would each like to buy a unit of the object). The buyers and sellers interact in order to trade units of the object in exchange for monetary payments. The economic theory of such models has been developed in Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), and Wilson (1985), in addition to a huge subsequent literature. See Fudenberg et al. (2007), Kadan (2007), or Araujo and de Castro (2009) for recent results. See Bolton and Dewatripont (2005, Chapter 7) for a textbook treatment. For monotonicity in the equilibrium strategies, see e.g., Chatterjee and Samuelson (1983, Theorem 1) and Satterthwaite and Williams (1989a, Definition of "regular" equilibrium) and Fudenberg et al. (2007, Theorem 1). The case of $N_s = 1 = N_b$ has seen particular attention, as models of bilateral trade.[24] The case of $N_s > 1$ and $N_b > 1$ has also seen particular attention, as "strategic" versions of supply and demand models, in which individual market participants do not act as competitive price takers. Although the theory literature has tended to treat these two cases separately, the identification strategy can accommodate both cases.

The valuation of player i for a unit of the object is the private information θ_i. The buyers announce "bid prices" and the sellers announce "ask prices" and trade proceeds. Suppose that $a_{(N_s)}$ is the N_s-th highest announcement and $a_{(N_s+1)}$ is the N_s+1-st highest announcement, both among the combined set of announcements (i.e., bids and asks) from buyers and sellers. Let $z(a) = ka_{(N_s)} + (1-k)a_{(N_s+1)}$ be the resulting transaction price, where $k \in [0,1]$ is a parameter of the model that might either be known or unknown by the econometrician (an example of a possibly incomplete specification of the model of the game). Then one possible allocation rule and transfer rule is

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } a_i > z(a) \\ p_i(a) & \text{if } a_i = z(a) \\ 0 & \text{if } a_i < z(a) \end{cases} \text{ and } \overline{t}_i(a) = \begin{cases} z(a) & \text{if i is a buyer and } a_i > z(a) \\ -z(a) & \text{if i is a seller and } a_i < z(a) \\ p_i(a)z(a) & \text{if i is a buyer and } a_i = z(a) \\ -(1-p_i(a))z(a) & \text{if i is a seller and } a_i = z(a) \\ 0 & \text{otherwise,} \end{cases}$$

where $p_i(a)$ reflects a tie-breaking rule with the condition that $\sum_{i=1}^{N} \overline{x}_i(a) = N_s$ for all a. In particular, in the case of $a_{(N_s)} > a_{(N_s+1)}$, the tie-breaking rule is such that $p_i(a) = 1$ when $a_i = z(a)$ and $k = 1$ and $p_i(a) = 0$ when $a_i = z(a)$ and $k = 0$. Therefore, ignoring ties by considering the situation that $a_{(N_s)} > a_{(N_s+1)}$, and because $a_{(N_s)} \geq z(a) \geq a_{(N_s+1)}$ with at least one inequality strict, the players with the N_s highest announcements, among both buyers and sellers, are allocated a unit of the object. The transaction price is $z(a)$, and buyers that are allocated a unit of the object pay $z(a)$ and sellers that are not allocated a unit of the object receive $z(a)$. See for example Fudenberg et al. (2007) for more details. These allocation and transfer rules might be unknown by the econometrician, if the econometrician does not know k, in which case the identification strategy involves identifying the allocation and transfer rules directly from the data.

[24]There are a variety of different "bilateral trade" or "bargaining" models, not all of which proceed in the same way. For example, Merlo and Tang (2012) study identification of a different bargaining model that evidently does not fit this allocation-transfer game framework.

The main assumption of the identification strategy is that the players use monotone strategies. For buyers, this requires that buyers announce that they are willing to pay relatively more for a unit of the object when their valuation for a unit of the object is relatively higher. For sellers, this requires that sellers announce that they require a relatively higher payment for a unit of the object when their valuation for a unit of the object is relatively higher. Further, equilibrium strategies can be difficult to characterize (e.g., Leininger et al. (1989) and Satterthwaite and Williams (1989a)), making it useful that assuming a property of the equilibrium is sufficient for the identification strategy, without needing to explicitly characterize the equilibrium solution. For example, in one particular case (with $k = 0$ and other assumptions), Satterthwaite and Williams (1989b) show that the equilibrium strategy for the buyers is the solution to a differential equation involving a combinatorial expression involving the unknown distribution of valuations.

Chatterjee and Samuelson (1983, Example 2) show in a specific example with $N_s = 1 = N_b$ that the strategy for the buyer or seller can involve a "flat spot" if the support of the distribution of valuations for the buyer is different from the support of the distribution of valuations for the seller, even with a continuous action space. Leininger et al. (1989) show that there exists equilibria in which both buyers and sellers use step functions as their strategies. One of these equilibria is particularly simple, with the valuations supported on $[0, 1]$. For some $\overline{\theta}$, a buyer with a valuation less than $\overline{\theta}$ bids 0 and a buyer with a valuation weakly greater than $\overline{\theta}$ bids $\overline{\theta}$. Conversely, a seller with a valuation weakly less than $\overline{\theta}$ asks $\overline{\theta}$ and a seller with a valuation greater than $\overline{\theta}$ asks 1. The corresponding *ex interim* expected allocation and *ex interim* expected transfer would not be differentiable.

C. Proofs

In order to economize on space, references to equations and quantities defined in the body of the paper are used in the proofs. The first result is a technical lemma used in the proof of Theorem 4.

Lemma 4. *Suppose that (Y, X) are random variables with $Y \in \mathbb{R}^d$ and $X \in \mathbb{R}$. Suppose $P(Y \in U|X = x_2) \geq P(Y \in U|X = x_1)$ for all Borel measurable upper sets U, for $x_2 \geq x_1$. Then, for any weakly increasing function $f(\cdot)$ and weakly increasing function $g(\cdot)$, $P(f(Y) \in U|g(X) = h_2) \geq P(f(Y) \in U|g(X) = h_1)$ for all Borel measurable upper sets U, for $h_2 \geq h_1$.*

Proof of Lemma 1. Obviously, Assumption 6**(a) implies Assumption 6**(b). By Assumption 4 (Correct beliefs), $\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i') = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i')$, because the distribution of $A_{-i}|\theta_i'$ is the same as the distribution of $a_{-i}(\theta_{-i})|\theta_i'$. Under the condition that $z_i \in \tilde{\mathcal{A}}_i(v_i)$ it holds that $v_i\overline{x}_i(z_i, a_{-i}) - \overline{t}_i(z_i, a_{-i})$ is a weakly decreasing function of a_{-i} given a_{-i} from the support by Assumption 6**(b), so $v_i\overline{x}_i(z_i, a_{-i}(\theta_{-i})) - \overline{t}_i(z_i, a_{-i}(\theta_{-i}))$ is a weakly decreasing function of θ_{-i} under the use of weakly increasing strategies by all players $j \in \mathcal{I}$.

Under Assumption 6**(c), by standard properties of affiliated random variables (e.g., Milgrom and Weber (1982, Theorem 5) or Milgrom (2004, Theorem 5.4.5)), $v_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i')$ is a weakly decreasing function of θ_i'. Leading to the same conclusion, under Assumption 6**(d), by standard properties of the usual multivariate stochastic order (e.g., Shaked and Shanthikumar (2007, Chapter 6)), it follows that $v_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i^{(1)}) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i^{(1)}) \geq v_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i^{(2)}) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i^{(2)})$ for $\theta_i^{(1)} \leq \theta_i^{(2)}$.

Assumption 6(a) follows by setting $\theta_i^{(1)} = \theta_i'$ and $\theta_i^{(2)} = \theta_i$ and $v_i = \theta_i$ and $z_i = a_i(\theta_i)$ in this inequality. This specification is allowed because, by assumption, $a_i(\theta_i) \in \tilde{\mathcal{A}}_i(\theta_i)$.

Assumption 6(b) follows from $\theta_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i) \geq \theta_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i'')$ for all $z_i \in \tilde{\mathcal{A}}_i(\theta_i)$, where the inequality is by the above inequality with $\theta_i^{(1)} = \theta_i$ and $\theta_i^{(2)} = \theta_i''$ and $v_i = \theta_i$. Using Assumption 6**(b), this implies that $\sup_{z_i \in \mathcal{A}_i}(\theta_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)) \geq \sup_{z_i \in \mathcal{A}_i}(\theta_iE_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i''))$. □

Proof of Lemma 2. By definition, $\overline{x}_i(a) = E(\tilde{x}_i(a)) = E(\tilde{x}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a) = E(\tilde{t}_i(a)) = E(\tilde{t}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$.

Consider $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$. Suppose that $a_i \in \mathcal{A}_i^d$ and $z_i' \in \mathcal{A}_i^d$. A point in the support of $A_{-i}|(A_i = z_i')$ combined with a point in the support of A_i is a point in the support of A, given the assumption on $\mathcal{A}^d$. Therefore, $\overline{x}_i(a_i, a_{-i})$ is point identified at all values used in the evaluation of $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$. The distribution of $A_{-i}|(A_i = z_i')$ is point identified since $z_i' \in \mathcal{A}_i^d$. Therefore, $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$ is point identified. It is similar for $E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$, $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i')$, and $E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$. Therefore, there is game-structure identification of differences per Definition 4. □

Proof of Theorem 1. The proof uses the fact that Assumption 6** implies Assumption 6 given the other conditions of the identification result, per Lemma 1.

By Assumption 3 (Optimal strategy is used), Equation 13 is a necessary condition for any action $\tilde{a}_i(\theta_i)$ used by player i. Then, under Assumption 4 (Correct beliefs), Equation 14 is an equivalent necessary condition. Then, under Assumptions 5 (Weakly increasing strategy is used), and 6

(Monotone effect of counterfactual beliefs on utility), Equation 18 is valid. Under Assumption 5 (Weakly increasing strategy is used), given that $z_i' < a_i(\theta_i) < z_i''$ are all used in the data, all elements of $\Theta_i(z_i')$ are less than all elements of $\Theta_i(a_i(\theta_i))$, and all elements of $\Theta_i(a_i(\theta_i))$ are less than all elements of $\Theta_i(z_i'')$, where $\Theta_i(\cdot)$ is defined in Equation 15. In particular, $\theta_i \in \Theta_i(a_i(\theta_i))$, all elements of $\Theta_i(z_i')$ are less than θ_i, and θ_i is less than all elements of $\Theta_i(z_i'')$. Then, combining Equations 16 and 17 with Equation 18, Equation 19 is valid. By Assumption 8 (Known bounds on actions), the valuation v_i associated with an observed action a_i must satisfy $a_{Li}(v_i) \leq a_i \leq a_{Ui}(v_i)$, and therefore $a_{Ui}^{-1}(a_i) \leq v_i \leq a_{Li}^{-1}(a_i)$ by construction of those functions. Equations 4, 6, 20 and 21 follow immediately, using Assumption 7 (Known bounds on valuations).

Now, for a given a_i, consider any $\tilde{\theta}_i < \Phi_{Li}(a_i')$ with $a_i' \leq a_i, a_i' \in \mathcal{A}_i^d$. If θ_i' is any valuation consistent with using action a_i', then $\theta_i' \geq \Phi_{Li}(a_i')$. Moreover, since $a_i' \in \mathcal{A}_i^d$ by construction, there is indeed some valuation θ_i' that uses action a_i'. By Assumption 5 (Weakly increasing strategy is used), the action used by valuation $\tilde{\theta}_i$ is weakly less than the action used by valuation $\theta_i' \geq \Phi_{Li}(a_i') > \tilde{\theta}_i$, so the action used by valuation $\tilde{\theta}_i$ is weakly less than a_i'. Moreover, since $\tilde{\theta}_i \ngeq \Phi_{Li}(a_i')$ by construction, valuation $\tilde{\theta}_i$ cannot use action a_i'. Consequently, player i with valuation $\tilde{\theta}_i$ must use an action strictly less than a_i'. By the contrapositive, any used action weakly greater than a_i' must correspond to a valuation weakly greater than $\Phi_{Li}(a_i')$. Consequently, because $a_i' \leq a_i$, the valuation θ_i corresponding to the use of action a_i must be weakly greater than $\Phi_{Li}(a_i')$. Since the above holds for any $a_i' \leq a_i, a_i' \in \mathcal{A}_i^d$, the valuation θ_i corresponding to the use of action a_i must be weakly greater than $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$. Consequently, $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ is a lower bound for the valuation corresponding to a_i. Similarly, $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$ is an upper bound for the valuation corresponding to a_i.

Therefore, considering the joint distribution of $(\theta_1, \theta_2, \ldots, \theta_{N_1})$ and corresponding observed actions $(A_1, A_2, \ldots, A_{N_1})$, it holds for all realizations that, for each $i \in \mathcal{J}$, $\Upsilon_{Li}(A_i) \leq \theta_i \leq \Upsilon_{Ui}(A_i)$. This is the characterization of the usual multivariate stochastic order mentioned in the discussion of Definition 3. □

Independent valuations. *Under Assumption 1*, the following adjustments are made to the proof. Under Assumption 1*, Equations 13 and 14 need not condition on θ_i since beliefs do not depend on valuation. Thus, Equations 9 and 11 are valid bounds for the valuation, even without Assumption 6 (Monotone effect of counterfactual beliefs on utility). Then, by arguments similar to those used previously in the proof of Theorem 1, the valuation corresponding to a_i must be between*

$\sup_{a_i' \le a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Li}(a_i')$ *and* $\inf_{a_i' \ge a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Ui}(a_i')$. *Thus, the valuation corresponding to* a_i *must be between* $\Gamma_{Li}(a_i)$ *and* $\Gamma_{Ui}(a_i)$ *defined in Equation 12.*

To establish game-structure identification of differences, $E_P(X_i|A_i = z_i) = E_P(\tilde{x}_i(A_i, A_{-i})|A_i = z_i) = E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i) = E_P(\overline{x}_i(z_i, A_{-i}))$, *where the first equality holds by definition of the game (and resulting allocations), the second equality holds by standard properties of conditioning and the law of iterated expectations (with respect to any randomness in the allocation), and the third equality holds because the actions of different players are independent. It is similar for* $E_P(T_i|A_i = z_i) = E_P(\bar{t}_i(z_i, A_{-i}))$. ★

Proof of Theorem 2. Arbitrarily choose an $a_i^* \in \mathcal{K}_i$, and specify that $E_P(\overline{x}_i(z_i, A_{-i})) = E_P(\overline{x}_i(a_i^*, A_{-i}))$ and $E_P(\bar{t}_i(z_i, A_{-i})) = E_P(\bar{t}_i(a_i^*, A_{-i}))$ for any $z_i \notin \mathcal{K}_i$, where the right sides are point identified from Assumption I of Theorem 2. Consequently, any such action z_i gives the same expected allocation and expected transfer as does the action a_i^*. Therefore, player i would get the same utility from action z_i as it would from action a_i^*. Consequently, for checking optimality of an action, it suffices to restrict to actions in $\mathcal{K}_i$.

For $i \in \mathcal{J}$, let $\Gamma_i(\cdot)$ defined on $\mathcal{A}_i^d$ be a strictly increasing function such that $\Gamma_{Li}(\cdot) \le \Gamma_i(\cdot) \le \Gamma_{Ui}(\cdot)$ from Assumption V of Theorem 2. For $i \notin \mathcal{J}$, if any, let $\Gamma_i(\cdot)$ be an arbitrary strictly increasing function on $\mathcal{A}_i^d$. This effectively implies that such players are "behavioral players," in the sense that subsequent steps of the analysis just rely on them behaving according to a certain distribution. This is consistent with "sharpness." Implicitly, this implies that such player i that uses action a_i is "assigned" to have a valuation $\Gamma_i(a_i)$.

Consider the distribution of actions according to conjectured strategies $\Gamma_i^{-1}(\cdot)$ defined on the support of $\Gamma_i(A_i)$ where $A_i \sim P(A)$. Since $\Gamma_i(\cdot)$ is strictly increasing on $\mathcal{A}_i^d$, $\Gamma_i^{-1}(\cdot)$ is strictly increasing on the support of $\Gamma_i(A_i)$. In particular, this implies Assumption 5 (Weakly increasing strategy is used) is satisfied. Thus, the distribution of actions is $(\Gamma_1^{-1}(\Gamma_1(A_1)), \Gamma_2^{-1}(\Gamma_2(A_2)), \ldots, \Gamma_N^{-1}(\Gamma_N(A_N))) = (A_1, A_2, \ldots, A_N)$, as claimed. Further, using Assumption IV of Theorem 2, the conjectured distribution of valuations has independent components, thus satisfying Assumption 1* (Independent valuations). Thus, in the analysis of utility maximization, it is not necessary to condition beliefs on valuation.

For $i \in \mathcal{J}$, by construction for given $a_i \in \mathcal{A}_i^d$, the corresponding valuation satisfies $a_{Ui}^{-1}(a_i) \le \Gamma_i(a_i) \le a_{Li}^{-1}(a_i)$ by Equations 9 and 11. Therefore, $a_{Li}(\Gamma_i(a_i)) \le a_i \le a_{Ui}(\Gamma_i(a_i))$. For the inequality $a_{Li}(\Gamma_i(a_i)) \le a_i$, if $a_{Li}^{-1}(a_i) = \infty$, then since $a_{Li}(\cdot)$ is weakly increasing, all v_i are such that $a_{Li}(v_i) \le a_i$.

So suppose that $a_{Li}^{-1}(a_i) < \infty$. Then because $a_{Li}(\cdot)$ is weakly increasing, $a_{Li}(\Gamma_i(a_i)) \leq a_{Li}(a_{Li}^{-1}(a_i)) \leq a_i$. The last inequality there holds by continuity of $a_{Li}(\cdot)$ by Assumption II of Theorem 2, which implies $a_{Li}(a_{Li}^{-1}(a_i)) = \lim_{t \to a_{Li}^{-1}(a_i)} a_{Li}(t)$. This sequence can be taken as elements of $\{v_i : a_{Li}(v_i) \leq a_i\}$ approaching $a_{Li}^{-1}(a_i)$; along that sequence, $a_{Li}(t) \leq a_i$, so $\lim_{t \to a_{Li}^{-1}(a_i)} a_{Li}(t) \leq a_i$. This set is non-empty since $\Gamma_i(a_i) \leq a_{Li}^{-1}(a_i)$, so the sup of the set is not $-\infty$. The inequality $a_i \leq a_{Ui}(\Gamma_i(a_i))$ is similar. This valuation $\Gamma_i(a_i)$ uses action a_i according to the strategies Γ_i^{-1}, from above. Thus, Assumption 8 (Known bounds on actions) is satisfied. It is obvious that Assumption 7 (Known bounds on valuations) is satisfied by construction, by Equations 9 and 11.

Consider the realization $(\Gamma_1(a_1), \Gamma_2(a_2), \ldots, \Gamma_N(a_N))$ for some $a \in \mathcal{A}^d$ from the distribution of valuations, which by construction uses the action a using the conjectured strategies. For each player $i \in \mathcal{J}$, the utility maximization problem is to maximize $\Gamma_i(a_i) E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i}))$. Specifying player i to have correct beliefs, thus satisfying Assumption 4 (Correct beliefs), whereby $\Pi_i(a_{-i}) = P(A_{-i})$ since the distribution of actions is the same as in the real data by the above, this is the same as maximizing $\Gamma_i(a_i) E_P(\overline{x}_i(z_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))$. The action that this valuation actually uses would satisfy the condition of utility maximization exactly when $\Gamma_i(a_i) E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{t}_i(a_i, A_{-i})) \geq \Gamma_i(a_i) E_P(\overline{x}_i(z_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))$ for all $z_i \in \mathcal{A}_i$. The following establishes this is true.

Consider $z_i \in \mathcal{A}_i$ such that $(a_i, z_i) \in \mathcal{R}_i^{\perp}$. By the setup in Assumption I of Theorem 2, this includes all $z_i \in \mathcal{K}_i$.

For $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) > 0\}$, $\Gamma_i(a_i) \geq \Gamma_{Li}(a_i) \geq \frac{E_P(\overline{t}_i(a_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))}$ by Equations 8, 9 and 12. This uses the condition that $a_i \in \mathcal{A}_i^d$ by construction, and the condition that $(a_i, z_i) \in \mathcal{R}_i^{\perp}$. Consequently, after re-arranging that inequality, the utility from action a_i weakly exceeds the utility from action z_i. Similarly, for $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) < 0\}$, $\Gamma_i(a_i) \leq \Gamma_{Ui}(a_i) \leq \frac{E_P(\overline{t}_i(a_i, A_{-i})) - E_P(\overline{t}_i(z_i, A_{-i}))}{E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i}))}$ by Equations 10 to 12. Consequently, after re-arranging that inequality, the utility from action a_i weakly exceeds the utility from action z_i.

For $z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) = 0\}$, by Assumption III of Theorem 2, it follows that $-E_P(\overline{t}_i(a_i, A_{-i})) \geq -E_P(\overline{t}_i(z_i, A_{-i}))$. Thus, the utility from action a_i weakly exceeds the utility from action z_i for any valuation of player i.

Therefore, Assumption 3 (Optimal strategy is used) is satisfied. □

Proof of Theorem 3. For the part of Theorem 3 about the assumptions: Assumption I of Theorem 2 holds by the last part of the independent valuations part of Theorem 1. Assumption II of Theorem 2 holds directly. Assumption III of Theorem 2 holds under the assumptions of Theorem 1, since $a_i \in \mathcal{A}_i^d$ implies that a_i maximizes utility for some valuation, which would only be consistent with $E_P(\overline{x}_i(a_i, A_{-i})) = E_P(\overline{x}_i(z_i, A_{-i}))$ if indeed $-E_P(\overline{t}_i(a_i, A_{-i})) \geq -E_P(\overline{t}_i(z_i, A_{-i}))$. Assumption IV of Theorem 2 holds because independent valuations implies independent actions.

For the part of Theorem 3 about existence of at least one specification of $\Gamma_i(\cdot)$: Under the true data generating process, per Assumption 5 (Weakly increasing strategy is used) there are weakly increasing strategies $a_i(\theta_i)$ that generate the data, for the true distribution of valuations. Then, let $\Gamma_i(a_i) = \text{sel}\Theta_i(a_i)$ defined on $a_i \in \mathcal{A}_i^d$, a selection from the set of valuations that uses action a_i per the discussion of Assumption 5 (Weakly increasing strategy is used). By the proof of Theorem 1, given the validity of the bounds, it must be that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$ on $\mathcal{A}_i^d$. Consider any pair $a_i \in \mathcal{A}_i^d$ and $a_i' \in \mathcal{A}_i^d$ with $a_i < a_i'$. Given the ordering of the sets $\Theta_i(\cdot)$ from Section 4.4, it must be that $\Gamma_i(a_i) < \Gamma_i(a_i')$, so $\Gamma_i(\cdot)$ is strictly increasing on $\mathcal{A}_i^d$.

For the next part of Theorem 3 about $\Gamma_i(\cdot)$: let $\Gamma_i(\cdot)$ defined on $\mathcal{A}_i^d$ be a strictly increasing function such that $\Gamma_{Li}(\cdot) \leq \Gamma_i(\cdot) \leq \Gamma_{Ui}(\cdot)$. Per the previous part of Theorem 3, at least one such function exists. Then let $\tilde{\Gamma}_i(\cdot) = \alpha\Gamma_i(\cdot) + (1-\alpha)\Gamma_{Li}(\cdot)$ for some $\alpha \in (0,1)$. Clearly, $\Gamma_{Li}(\cdot) \leq \tilde{\Gamma}_i(\cdot) \leq \Gamma_{Ui}(\cdot)$. Moreover, clearly $\tilde{\Gamma}_i(\cdot)$ is strictly increasing because $\Gamma_i(\cdot)$ is strictly increasing and $\Gamma_{Li}(\cdot)$ is weakly increasing. Further, $0 \leq \tilde{\Gamma}_i(\cdot) - \Gamma_{Li}(\cdot) = \alpha\left(\Gamma_i(\cdot) - \Gamma_{Li}(\cdot)\right) \leq \alpha\left(\Theta_{Ui} - \Theta_{Li}\right)$, so $\sup_{a_i \in \mathcal{A}_i^d}\left(\tilde{\Gamma}_i(a_i) - \Gamma_{Li}(a_i)\right) < \epsilon$ by taking $\alpha < \frac{\epsilon}{\Theta_{Ui} - \Theta_{Li}}$. Similar arguments based on $\tilde{\Gamma}_i(\cdot) = \alpha\Gamma_i(\cdot) + (1-\alpha)\Gamma_{Ui}(\cdot)$ establish that $0 \leq \sup_{a_i \in \mathcal{A}_i^d}\left(\Gamma_{Ui}(a_i) - \tilde{\Gamma}_i(a_i)\right) < \epsilon$.

The part of Theorem 3 about distributional properties: $(\Gamma_1(A_1), \Gamma_2(A_2), \ldots, \Gamma_{N_1}(A_{N_1}))$ is the same as $(\Gamma_1(a_1(\theta_1)), \Gamma_2(a_2(\theta_2)), \ldots, \Gamma_{N_1}(a_{N_1}(\theta_{N_1})))$, where $a_i(\cdot)$ is weakly increasing per Assumption 5. □

Proof of Theorem 4. Arbitrarily choose $a_i^* \in \mathcal{K}_i^i$. For any $(z_i, a_{-i}) \notin \mathcal{K}_i$ such that $z_i \notin \mathcal{K}_i^i$, specify that $\overline{x}_i(z_i, a_{-i}) = \overline{x}_i(a_i^*, a_{-i})$ and $\overline{t}_i(z_i, a_{-i}) = \overline{t}_i(a_i^*, a_{-i})$, for all $a_{-i} \in \mathcal{A}_{-i}^d$. In these specifications, by Assumption I of Theorem 4, the right sides are point identified. Given the (subsequent) expressions for the utility maximization problem, this implies that a player i would get the same utility from action z_i as it would from action a_i^*. Consequently, for checking for the maximal amount of foregone utility of an action, it will suffice to restrict attention to actions in $\mathcal{K}_i^i$. The allocation rule and transfer rule for $a_{-i} \notin \mathcal{A}_{-i}^d$ is irrelevant, so can be specified arbitrarily.

Except for the part about independent components, the second and third paragraphs of the proof of Theorem 2 remain true after substituting Υ for Γ.

Consider the realization $(\Upsilon_1(a_1), \Upsilon_2(a_2), \ldots, \Upsilon_N(a_N))$ for some $a \in A^d$ from the distribution of valuations, which by construction uses the action a using the conjectured strategies $\Upsilon_i^{-1}(\cdot)$. For each player $i \in \mathcal{J}$, the utility maximization problem is to maximize $\Upsilon_i(a_i)E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\tilde{\theta}_i = \Upsilon_i(a_i)) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\tilde{\theta}_i = \Upsilon_i(a_i))$, where the $\tilde{\theta}$ notation reflects the conjectured valuation, which may not equal the "true" valuations in the data. Specifying player i to have correct beliefs, thus satisfying Assumption 4, $\Pi_i(a_{-i}|\tilde{\theta}_i = t) = \Pi_i(\Upsilon_{-i}^{-1}(\tilde{\theta}_{-i})|\tilde{\theta}_i = t) = \Pi_i(\Upsilon_{-i}^{-1}(\Upsilon_{-i}(A_{-i}))|\Upsilon_i(A_i) = t) = P(A_{-i}|\Upsilon_i^{-1}(\Upsilon_i(A_i)) = \Upsilon_i^{-1}(t)) = P(A_{-i}|A_i = \Upsilon_i^{-1}(t))$. The first equality is the definition of correct beliefs in this setup, the second equality uses the construction of $\tilde{\theta}$, and the third and fourth equalities use that Υ_i^{-1} is strictly increasing on the support of $\Upsilon_i(A_i)$. Thus, utility maximization is the same as maximizing $\Upsilon_i(a_i)E_P(\overline{x}_i(z_i, A_{-i})|A_i = \Upsilon_i^{-1}(\Upsilon_i(a_i))) - E_P(\overline{t}_i(z_i, A_{-i})|A_i = \Upsilon_i^{-1}(\Upsilon_i(a_i)))$, which is the same as maximizing $\Upsilon_i(a_i)E_P(\overline{x}_i(z_i, A_{-i})|A_i = a_i) - E_P(\overline{t}_i(z_i, A_{-i})|A_i = a_i)$.

The arguments for Assumptions 5, 7 and 8 being satisfied are the same as in the proof of Theorem 2.

Consider a given $a_i \in \mathcal{A}_i^d$, and consider another $z_i \in \mathcal{K}_i^i$. Consider the foregone utility that player i with valuation $\Upsilon_i(a_i)$ gets from action a_i compared to from action z_i. Consider any $z_i' < a_i < z_i''$ with $\{z_i', z_i''\} \in \mathcal{A}_i^d$; if none exist, then the upper bound on foregone utility comparing action a_i to action z_i is ∞. Under the conditions of Theorem 4(b), it suffices to restrict attention to the actions in $\tilde{\tilde{\mathcal{A}}}_i$.

Suppose that $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0$. Thus, $\Upsilon_i(a_i) \geq \Upsilon_{Li}(a_i) \geq \frac{E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')}{E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')}$ by Equations 3, 4 and 7. This uses the condition that $a_i \in \mathcal{A}_i^d$ by construction. This also use the fact that $(a_i, z_i, z_i', z_i'') \in \mathcal{R}_i$, which is true by construction.

After re-arranging that inequality, $\Upsilon_i(a_i)\Big[E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')\Big] - \Big[E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')\Big] \geq 0$. Equivalently, $\Upsilon_i(a_i)\Big[\chi_i(a_i, a_i) - \chi_i(a_i, a_i) + \chi_i(a_i, z_i') - \chi_i(z_i, a_i) + \chi_i(z_i, a_i) - \chi_i(z_i, z_i'')\Big] - \Big[\tau_i(a_i, a_i) - \tau_i(a_i, a_i) + \tau_i(a_i, z_i') - \tau_i(z_i, a_i) + \tau_i(z_i, a_i) - \tau_i(z_i, z_i'')\Big] \geq 0$. Thus, $\Upsilon_i(a_i)\Big[\chi_i(a_i, a_i) - \chi_i(z_i, a_i)\Big] - \Big[\tau_i(a_i, a_i) - \tau_i(z_i, a_i)\Big] \geq \Upsilon_i(a_i)\Big[\chi_i(a_i, a_i) - \chi_i(a_i, z_i') - [\chi_i(z_i, a_i) - \chi_i(z_i, z_i'')]\Big] - \Big[\tau_i(a_i, a_i) - \tau_i(a_i, z_i') - [\tau_i(z_i, a_i) - \tau_i(z_i, z_i'')]\Big]$. Therefore, the amount of foregone utility is no more than $-\Big(\Upsilon_i(a_i)\Big[\chi_i(a_i, a_i) - \chi_i(a_i, z_i') - [\chi_i(z_i, a_i) - \chi_i(z_i, z_i'')]\Big] - \Big[\tau_i(a_i, a_i) - \tau_i(a_i, z_i') - [\tau_i(z_i, a_i) - \tau_i(z_i, z_i'')]\Big]\Big)$.

Suppose that $E_P(\bar{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\bar{x}_i(z_i, A_{-i})|A_i = z_i'') < 0$. Thus, $\Upsilon_i(a_i) \leq \Upsilon_{Ui}(a_i) \leq \frac{E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i') - E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')}{E_P(\bar{x}_i(a_i,A_{-i})|A_i=z_i') - E_P(\bar{x}_i(z_i,A_{-i})|A_i=z_i'')}$ by Equations 5 to 7. Consequently, after re-arranging that inequality, the same bound as above obtains.

Therefore, an upper bound for the foregone utility comparing action a_i and z_i is $\inf_{\{z_i',z_i''\}\in\mathcal{Z}_i(a_i,z_i)} \Big(-\big(\Upsilon_i(a_i)\big[\chi_i(a_i,a_i)-\chi_i(a_i,z_i')-[\chi_i(z_i,a_i)-\chi_i(z_i,z_i'')]\big]-\big[\tau_i(a_i,a_i)-\tau_i(a_i,z_i')-[\tau_i(z_i,a_i)-\tau_i(z_i,z_i'')]\big]\big)\Big)$, where $\mathcal{Z}_i(a_i, z_i) = \{\{z_i', z_i''\} \in \mathcal{A}_i^d : z_i' < a_i < z_i'' \text{ and } \chi_i(a_i, z_i') \neq \chi_i(z_i, z_i'')\}$. And, therefore the overall upper bound on the foregone utility of a player with valuation $\Upsilon_i(a_i)$ who uses action a_i is the sup of this over $z_i \in \mathcal{K}_i^i$ and $z_i \neq a_i$, and $z_i \in \tilde{\tilde{\mathcal{A}}}_i$ under the conditions of Theorem 4(b).

For Theorem 4(c): The conditions of Lemma 1 are satisfied, by the following arguments. Since all players use weakly increasing strategies in the data by assumption, including any $i \notin \mathcal{J}$, by the same arguments as before, any distributional property of $F(\theta)$ that is preserved by weakly increasing component-wise transformations is also a property of $(\Upsilon_1(A_1), \Upsilon_2(A_2), \ldots, \Upsilon_N(A_N))$. Therefore, if Assumption 6**(c) is true about $F(\theta)$, then $(\Upsilon_1(A_1), \Upsilon_2(A_2), \ldots, \Upsilon_N(A_N))$ is affiliated, since affiliation is preserved under weakly increasing transformations (e.g., Milgrom and Weber (1982, Theorem 3)). Alternatively, if Assumption 6**(d) is true about $F(\theta)$, then it is also true about $(\Upsilon_1(A_1), \Upsilon_2(A_2), \ldots, \Upsilon_N(A_N))$, because the property in Assumption 6**(d) is preserved under weakly increasing transformations by Lemma 4. Assumption 6**(a) is true directly by the assumption for the true allocation and transfer rules. By the above derivations, players $i \in \mathcal{J}$ have correct beliefs and all players use a monotone strategy. For the assumption of Lemma 1 on $\Upsilon_i^{-1}(v_i) \in \tilde{\mathcal{A}}_i(v_i)$, it has been established that $a_{Li}(\Upsilon_i(a_i)) \leq a_i \leq a_{Ui}(\Upsilon_i(a_i))$ for all $a_i \in \mathcal{A}_i^d$, which by the assumption of this result implies that $\Upsilon_i^{-1}(v_i) \in \tilde{\mathcal{A}}_i(v_i)$ for every v_i that arises of the form $v_i = \Upsilon_i(a_i)$ with $a_i \in \mathcal{A}_i^d$. Thus, by Lemma 1, Assumption 6 is satisfied. □

Proof of Theorem 5. The corresponding parts of the proof of Theorem 3 remain true after substituting Υ for Γ. □

Proof of Theorem 6. From Assumptions 9 (Continuous action space and no point masses in distribution of actions), 11 (Differentiable *ex interim* expected allocation and expected transfer), 12 (Game-structure identification of derivatives), and 13 (Non-zero marginal expected allocation), let $\mathcal{E}_i = (\text{int}(\mathcal{A}_i))^C \cup \mathcal{E}_{i,d} \cup \mathcal{E}_{i,r} \cup \mathcal{E}_{i,m}$ and $\mathcal{E} = \cup_{i\in\mathcal{J}}(\mathcal{E}_i \times \mathcal{A}_{-i})$, the set of a with at least one component an element of some $\mathcal{E}_i$ with $i \in \mathcal{J}$. Equivalently, $\mathcal{E}^C = \cap_{i\in\mathcal{J}}(\mathcal{E}_i \times \mathcal{A}_{-i})^C$; thus, if

$a \in \mathcal{E}^C$, then $a_i \in \mathcal{E}_i^C$ for all players $i \in \mathcal{J}$. Only for this part of this proof, use the notation that $\hat{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N_1})$ and $\hat{A} = (A_1, A_2, \ldots, A_{N_1})$. It follows that $P(\hat{A} \in \mathcal{E}) = 0$. Then $P(\hat{\theta} \in B) = P(\hat{\theta} \in B, \hat{A} \in \mathcal{E}^C) + P(\hat{\theta} \in B, \hat{A} \in \mathcal{E}) = P(\hat{\theta} \in B, \hat{A} \in \mathcal{E}^C) = P(\hat{\theta} \in B | \hat{A} \in \mathcal{E}^C)$ for any Borel set B, so it is enough to restrict the identification problem to recovering the distribution of $\hat{\theta}$ from actions in $\mathcal{E}^C$. By Assumptions 2 (Action space is ordered), 3 (Optimal strategy is used), 9 (Continuous action space and no point masses in distribution of actions), and 11 (Differentiable *ex interim* expected allocation and expected transfer), Equation 29 is the necessary condition for any action used by player $i \in \mathcal{J}$ in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C$:

$$\theta_i \left. \frac{\partial E_{\Pi_i}(\bar{x}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} - \left. \frac{\partial E_{\Pi_i}(\bar{t}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} = 0. \tag{29}$$

By Assumptions 1 (Dependent valuations), 5 (Weakly increasing strategy is used), 9 (Continuous action space and no point masses in distribution of actions), and 10 (Smooth distribution of valuations), conditioning on θ_i is equivalent to conditioning on $A_i = a_i(\theta_i)$, because if two distinct valuations use the same action the entire interval between those valuations would also use the same action, resulting in a point mass in the distribution of actions by Assumption 10 (Smooth distribution of valuations). So by Assumption 4 (Correct beliefs), Equation 30 is valid for actions in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C$:

$$\theta_i \left. \frac{\partial E_P(\bar{x}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} - \left. \frac{\partial E_P(\bar{t}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} = 0. \tag{30}$$

Under Assumption 13 (Non-zero marginal expected allocation), Equation 31 is valid for all actions used by player $i \in \mathcal{J}$ in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C$:

$$\theta_i = \Psi_i(A_i). \tag{31}$$

By Assumption 12 (Game-structure identification of derivatives), $\Psi_i(a_i)$ is point identified for all $a_i \in \mathcal{A}_i^d \cap \text{int}(\mathcal{A}_i) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C \cap \mathcal{E}_{i,r}^C$. □

Independent valuations. *Under Assumption 1*, the following adjustments are made to the proof. Equation 29 need not condition on θ_i since beliefs are independent of valuation. Similarly, Equation 30 is valid without conditioning on A_i.* ★

Proof of Lemma 3. The definitions of $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$ are given in Equation 23. Therefore, by substitution, the expressions in Equation 28 are valid. Let $a_i \in \mathcal{A}_i^d$ be given, and let $\mathcal{S}$ be given with the stated properties. Let $a_i' \in \mathcal{A}_i^d \cap \mathcal{S}$. By assumption, $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ are point identified for all a_{-i} in a probability 1 subset of the support of $A_{-i}|(A_i = a_i)$. Therefore, given that the distribution of $A_{-i}|(A_i = a_i)$ is point identified by assumption, $E_P(E_P(X_i|A_i = a_i', A_{-i})|A_i = a_i)$ and $E_P(E_P(T_i|A_i = a_i', A_{-i})|A_i = a_i)$ are point identified. Consequently, the existence and values of $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified by the existence and values of the limits corresponding to expressions in Equation 28. □

Proof of Lemma 4. The claim is trivial for $h_2 = h_1$, so consider $h_2 > h_1$. $P(Y \in U|g(X) = h_0) = P(Y \in U|X \in g^{-1}(h_0)) = \int P(Y \in U|X = x)dP(X = x|X \in g^{-1}(h_0)) \in [\int \inf_{x\in g^{-1}(h_0)} P(Y \in U|X = x)dP(X = x|X \in g^{-1}(h_0)), \int \sup_{x\in g^{-1}(h_0)} P(Y \in U|X = x)dP(X = x|X \in g^{-1}(h_0))]$. Since $g(\cdot)$ is weakly increasing, if $x_1 \in g^{-1}(h_1)$ then $g(x_1) = h_1 < h_2$ so if $g(x_2) = h_2$ it must be that $x_2 \geq x_1$ (since $x_2 < x_1$ would imply $g(x_2) \leq g(x_1)$), so $x_1 \leq \inf\{x_2 : x_2 \in g^{-1}(h_2)\}$. Therefore, any value of $P(Y \in U|X = x)$ where $x \in g^{-1}(h_1)$ is less than or equal to all values of $P(Y \in U|X = x)$ where $x \in g^{-1}(h_2)$. Therefore, $\inf_{x\in g^{-1}(h_2)} P(Y \in U|X = x) \geq \sup_{x\in g^{-1}(h_1)} P(Y \in U|X = x)$. Therefore, $P(Y \in U|g(X) = h_2) \geq P(Y \in U|g(X) = h_1)$. This implies by Shaked and Shanthikumar (2007, Theorem 6.B.16) that $P(f(Y) \in U|g(X) = h_2) \geq P(f(Y) \in U|g(X) = h_1)$. □

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